Detailed Solution to 2007 Final

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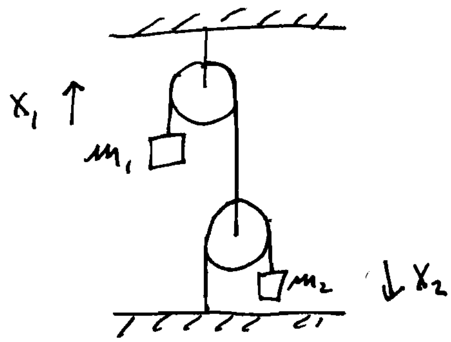
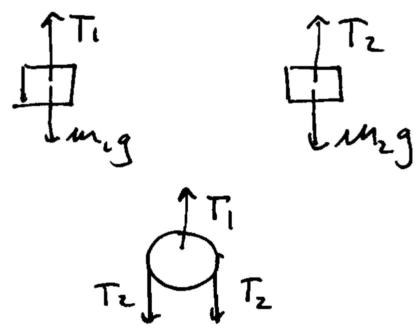
1) Double Atwood. See K & K pg 75 & 104



FBDs



a) If m_2 moves down x_2 then m_1 moves up $x_2/2$

Thus $x_1 = x_2/2$ or $\ddot{x}_1 = \ddot{x}_2/2$ (The sign depends on coords.)

b)
$$\left.\begin{aligned} m_1 \ddot{x}_1 &= T_1 - m_1 g \\ m_2 \ddot{x}_2 &= m_2 g - T_2 \\ T_1 &= 2T_2 \end{aligned}\right\} \qquad \begin{aligned} m_1 a_1 &= T_1 - m_1 g \\ 2 m_2 a_2 &= 2 m_2 g - 2T_2 \end{aligned}$$

or $(2m_2)(2a_1) + m_1 a_1 = 2m_2 g - m_1 g$

$$\Rightarrow \quad a_1 = \frac{(2m_2 - m_1) g}{m_1 + 4m_2}$$

2) In the limit that $m_b \ll m_w$ the cm does not change. Both $\vec{p}$ & $\vec{L}$ are conserved but ~~the~~ rise in height is determined only by consv of $\vec{p}$.

$$m_b V_b = (m_b + m_w) V_w \quad \Rightarrow \quad V_w = \left(\frac{m_b}{m_b + m_w}\right) V_b \rightarrow \frac{m_b}{m_w} V_b$$

$$(m_b + m_w) g h = \frac{1}{2}\left(\frac{m_b}{m_b + m_w}\right)^2 V_b^2 (m_b + m_w)$$

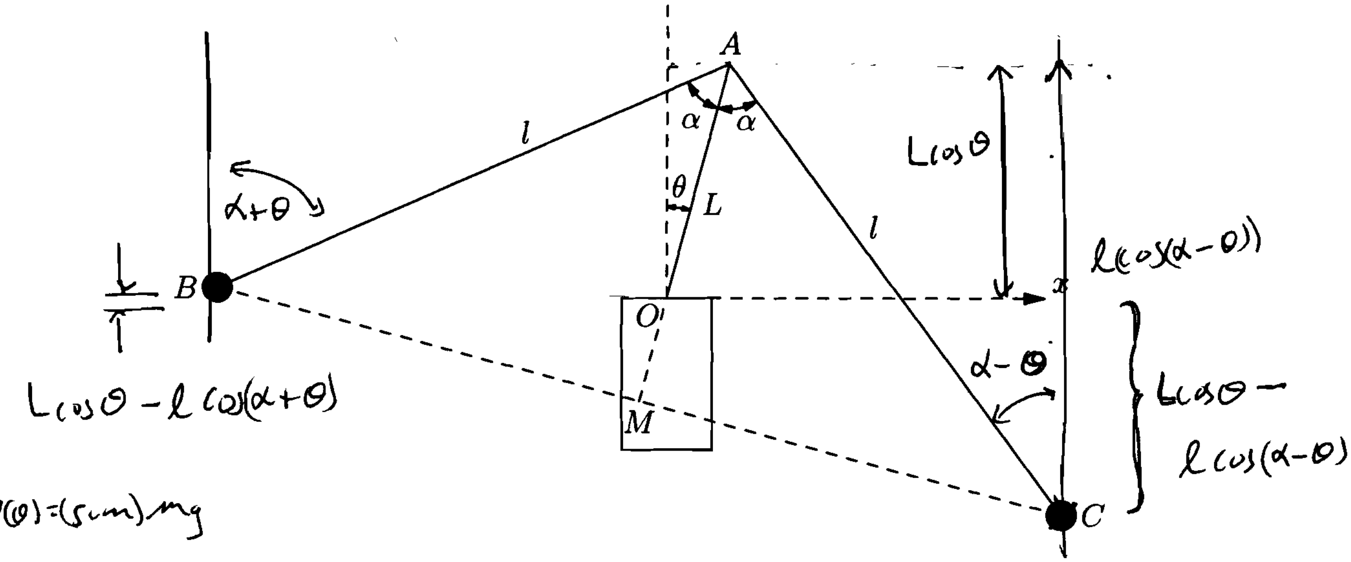
$$\Rightarrow h = \frac{V_b^2}{2g}\left(\frac{m_b}{m_b + m_w}\right)^2 \rightarrow \frac{V_b^2}{2g}\left(\frac{m_b}{m_w}\right)^2$$

3. (a) The potential energy $U(\theta)$ is given by   Kik pg 175

$$U(\theta) = mg(y_1 + y_2)$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the two masses in the coordinate system shown in Figure 1. Now, instead of calculating y_1 and y_2, let us use a trick.

A second method for the geometry:



$\Rightarrow U(\theta) = (\text{sum})mg$

Figure 1: Geometry of the teeter toy.

Notice how the ABC triangle in Figure 1 is isosceles, and notice that $y_1 + y_2 = 2y_{\text{avg}}$, where y_{avg} is the y-coordinate of the middle of the BC segment; let's call this point M. Now, AO bisects the angle BAC, and since ABC is isosceles, AO must intersect BC at its midpoint, M. Thus O lies on AM, and we have

$$OM = l\cos\alpha - L \qquad y_{\text{avg}} = -OM\cos\theta$$

and thus

$$U(\theta) = -2mg\cos\theta(l\cos\alpha - L) \tag{1}$$

Obviously, the same result could have been obtained by calculating what y_1 and y_2 are; by looking carefully at Figure 1, we see that

$$y_{1,2} = L\cos\theta - l\cos(\alpha \mp \theta)$$

and thus

$$\begin{aligned} y_1 + y_2 &= 2L\cos\theta - l\big(\cos(\alpha-\theta) + \cos(\alpha+\theta)\big) = 2L\cos\theta - 2l\cos\alpha\cos\theta \\ &= 2\cos\theta(L - l\cos\alpha) \end{aligned}$$

where I used $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$.
Both methods give us

$$\boxed{U(\theta) = 2mg\cos\theta(L - l\cos\alpha)} \tag{2}$$

(b) We can find the equilibrium positions by searching for the zeroes of the derivative $dU/d\theta$:

$$\begin{aligned} 0 \overset{!}{=} \frac{dU}{d\theta} &= 2mg\sin\theta(l\cos\alpha - L) \\ &\implies \sin\theta = 0 \end{aligned}$$

and since $\theta = \pi$ corresponds to the teeter toy staying upside-down, *i.e.*, going through its support, we can discard that solution and conclude that the only equilibrium position is

$$\boxed{\theta_{\text{eq}} = 0} \tag{3}$$

(Do note that if $L = l\cos\alpha$, the system is in equilibrium at any angle; this is because the center of mass of the teeter toy coincides with the pivot point in this case.)

(c) The equilibrium is stable if the potential energy has a minimum at that point. A sufficient condition is

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=\theta_{\text{eq}}=0} > 0 \implies l\cos\alpha > L$$

where we used the fact that m and g are positive. Looking back at Figure 1, we see that this condition requires the center of mass of the teeter toy to be below its pivot point, which is what we would expect (a pendulum has its center of mass below its pivot, and it is stable; try to balance a pencil on its tip — it is not stable; its center of mass is above the pivot.)
Again, the condition for stable equilibrium is

$$\boxed{l\cos\alpha > L} \tag{4}$$

(d) The energy of the oscillator is

$$E = K + U = \frac{1}{2}B\dot{q}^2 + \frac{1}{2}Aq^2 + \text{const.}$$

Differentiating this with respect to time, and using the fact that total energy is conserved, *i.e.*, $dE/dt = 0$,

$$B\dot{q}\ddot{q} + Aq\dot{q} = 0$$

and dividing by $\dot{q}$,[1]

$$B\ddot{q} + Aq = 0 \implies \ddot{q} + \frac{A}{B}q = 0$$

which is the well-known simple harmonic motion equation, with angular frequency given by

$$\boxed{\omega^2 = \tfrac{A}{B}} \tag{5}$$

To calculate the small oscillations frequency for the teeter toy, we need its kinetic energy. We can again use a trick: as suggested in the hint, note that the masses are moving on circles centered at the pivot point. The reason is that the toy is a rigid body, and thus the distance between any two points is constant; in particular, the distance between the pivot and each of the masses is always equal to (see Figure 1):

$$\begin{aligned} OB = OC &= \sqrt{OM^2 + MB^2} = \sqrt{(l\cos\alpha - L)^2 + l^2\sin^2\alpha} \\ &= \sqrt{l^2 + L^2 - 2lL\cos\alpha} \end{aligned}$$

The kinetic energy is then simply

$$K = \frac{1}{2}2m\,(OB^2)\,\dot{\theta}^2 = m\dot{\theta}^2(l^2 + L^2 - 2lL\cos\alpha)$$

Remember the potential energy

$$U = 2mg\cos\theta(L - l\cos\alpha)$$

which for small θ becomes ($\cos\theta \approx 1 - \theta^2/2$)

$$U \approx 2mg(L - l\cos\alpha) + mg(l\cos\alpha - L)\theta^2 = mg(l\cos\alpha - L)\theta^2 + \text{const.}$$

and thus

$$\begin{aligned} \omega^2 &= \frac{2mg(l\cos\alpha - L)}{2m(l^2 + L^2 - 2lL\cos\alpha)} \\ &= \frac{g(l\cos\alpha - L)}{l^2 + L^2 - 2lL\cos\alpha} \end{aligned}$$

[1] Of course, $\dot{q}$ could vanish at some points. However, unless the oscillator is at rest, $\dot{q}$ will only vanish at isolated points, and at those points we can reobtain the solution by considerations of continuity and differentiability.

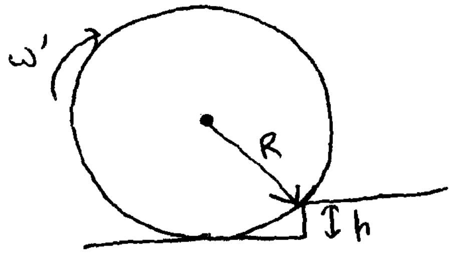
So the angular frequency of small oscillations of the teeter toy is

$$\boxed{\omega = \sqrt{\frac{g(l\cos\alpha - L)}{l^2 + L^2 - 2lL\cos\alpha}}}$$

Note that if $\alpha = 0$, this reduces to $\sqrt{g/(l-L)}$, which is what you would expect, since then the teeter toy looks just like a pendulum of length $l - L$.

4a) $L = L_{orb} + L_{spin} = mv_0(R-h) + I\omega = mv_0(R-h) + \frac{2}{5}mR^2\left(\frac{v}{R}\right) = \left(\frac{7}{5}R - h\right)mv_0$

b)



$\parallel$-axis thm.

$I_{pivot} = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$

cons. of L during "collision" at pivot point

$$L_{init} = L_{final} = \left(\frac{7}{5}R - h\right)mv_0 = \frac{7}{5}mR^2\omega'$$

$$\omega' = \frac{v_0}{R}\left(1 - \frac{5h}{7R}\right)$$

cons. of E $\quad \frac{1}{2}\left(\frac{7}{5}mR^2\right)\omega'^2 = \frac{7}{10}mv_0^2\left(1 - \frac{5h}{7R}\right)^2 = mgh$

$$v_0 = \sqrt{\frac{10}{7}gh}\left(\frac{1}{1 - \frac{5h}{7R}}\right)$$

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5. a) $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 - \frac{k}{r^\alpha}$

common mistakes

1) $\dot{r} = 0$

2) $\dot{r} = -2R\sin\theta$

$r = 2R\cos\theta, \quad \dot{r} = -2R\sin\theta\,\frac{d\theta}{dt}$

$$E = 2mR^2\sin^2\theta\,\dot{\theta}^2 + 2mR^2\cos^2\theta\,\dot{\theta}^2 - k/r^\alpha = 2mR^2\dot{\theta}^2 - k/r^\alpha$$

$\ell = mrv_t = mr(r\dot{\theta})$, so $\dot{\theta} = \ell/mr^2$

substitute in; $\quad E = \frac{2R^2\ell^2}{mr^4} - \frac{k}{r^\alpha} = \text{const.}$

Know that E, ℓ are conserved. This can only hold if $E=0$ given the above equation

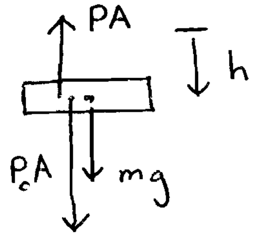
b) $E = 0$ requires $\alpha = 4$ and $k = \frac{2R^2\ell^2}{m}$ $\left(\text{Can also set } \frac{dE}{dr} = 0\right)$

c) From above, $\ell = \sqrt{\frac{km}{2R^2}}$

6) a) (3 points) The process is isothermal. Thus

$$P_i V_i = P_f V_f \quad \Rightarrow \quad P_0 V = P(V - Ah_0) \qquad (1)$$

(3 points)



$$m\ddot{h} = mg + P_0 A - PA$$

At equilibrium $\ddot{h} = 0 \Rightarrow P = P_0 + mg/A \qquad (2)$

(If we assume at this step that $mg \ll AP_0$ to set $P_0 = P_i$ then we will find $h_0 = 0$. The question must be asking for the small but nonzero value of h_0 that is leading order in mg/AP_0.)

(4 points) We put (1) and (2) together

$$P_0 V = (P_0 + mg/A)(V - Ah_0)$$

$$\Rightarrow \quad Ah_0 = V - \frac{P_0 V}{P_0 + mg/A}$$

$$\Rightarrow \quad h_0 = \frac{V}{A}\left(1 - \frac{P_0}{P_0 + mg/A}\right)$$

$$\Rightarrow \quad h_0 \approx \frac{mgV}{P_0 A^2} \qquad (\text{using } mg \ll AP_0)$$

Note $mg \ll AP_0 \Rightarrow V - Ah_0 \approx V$

b) (3 points) The process is adiabatic. Thus

See also PS #10

$$P_i V_i^{\gamma} = P_f V_f^{\gamma}$$

$$\Rightarrow \left(P_0 + {}^{mg}/_{A}\right)\left(V - Ah_0\right)^{\gamma} = P(x)\left(V - Ah_0 - Ax\right)^{\gamma}$$

where $h = h_0 + x$

(3 points) We have from (a)

$$m\ddot{x} = mg + P_0 A - P(x) A$$

We need to reduce this eqn. to the form

$$m\ddot{x} = -kx$$

from which we can deduce $\omega = \sqrt{\frac{k}{m}}$.

(4 points)

$$m\ddot{x} \approx \underbrace{mg + P_0 A - P(0)A}_{\text{we know the } 1^{st} \text{ 3 terms vanish}} - x P'(0) A + \mathcal{O}(x^2)$$

From part (a), we know the 1st 3 terms vanish

$$P(x) = \left(P_0 + {}^{mg}/_{A}\right)\left(\frac{V - Ah_0}{V - Ah_0 - Ax}\right)^{\gamma}$$

$$\approx P_0 \left(\frac{1}{1 - \frac{A}{V}x}\right)^{\gamma} \qquad \text{using } mg \ll AP_0$$

$$P'(x) \approx \frac{\gamma P_0 A}{V}\left(\frac{1}{1 - \frac{A}{V}x}\right)^{\gamma+1}$$

$$P'(0) \approx \frac{\gamma P_0 A}{V} \Rightarrow k \approx \frac{\gamma P_0 A^2}{V} \Rightarrow \omega \approx \sqrt{\frac{\gamma P_0 A^2}{mV}}$$

c) (no points)

$$\omega \approx \left[\frac{\left(\frac{7}{5}\right)\left(10^{6}\,\mathrm{dyn\ cm^{-2}}\right)\left(\frac{\pi}{4}\,\mathrm{cm^{2}}\right)^{2}}{\left(10^{-3}\,\mathrm{g}\right)\left(5600\,\mathrm{cm^{3}}\right)} \right]^{1/2}$$

$$4000\ \mathrm{cm^{3}}$$

$$= \left[\left(\frac{\pi}{8}\right)^{2} 10^{6}\ \mathrm{s^{-2}} \right]^{1/2}$$

$$= \frac{\pi}{8}\, 10^{3}\ \mathrm{Hz}$$

$$\nu = \omega/2\pi = \frac{1}{16}\, 10^{3}\ \mathrm{Hz} = 67.5\ \mathrm{Hz}$$

7) a) The farmer sees the pole Lorentz contracted by an amount

$$l = l_0 \sqrt{1 - v^2/c^2}$$

We require the pole fit in the barn $l = \frac{4}{5} l_0$

$$\Rightarrow \quad \frac{4}{5} = \sqrt{1 - v^2/c^2} \qquad \left(\gamma = \frac{5}{4}\right)$$

$$\Rightarrow \quad \frac{v}{c} = \frac{3}{5}$$

b) event A: back of pole passes door A

event B: front of pole passes door B

(t, x): farmer's frame

(t', x'): vaulter's frame

Choose the coordinate system such that

$$t_A = x_A = t'_A = x'_A = 0$$

In the farmer's frame

$$t_B = 0 \qquad x_B = \frac{4}{5} l_0$$

We use a Lorentz transformation

$$t'_B = \gamma \left(t_B - \frac{x_B v}{c^2}\right) = \frac{5}{4}\left(0 - \frac{3}{5}\,\frac{4}{5}\,\frac{l_0}{c}\right)$$
$$= -\frac{3}{5}\,\frac{l_0}{c}$$

event B happens before event A in vaulter's frame!

c) event C : back of pole passes door B

In the farmer's frame

$$t_c = \frac{(4/5)\,l_0}{(3/5)\,c} = \frac{4}{3}\frac{l_0}{c} \qquad x_c = \frac{4}{5}l_0$$

Again using the Lorentz transformation

$$t_c' = \gamma\left(\frac{4}{3}\frac{l_0}{c} - \frac{4}{5}\frac{3}{5}\frac{l_0}{c}\right) \qquad \gamma = \frac{5}{4}$$

$$= \frac{l_0}{c}\left(\frac{5}{3} - \frac{3}{5}\right)$$

$$= \frac{16}{15}\frac{l_0}{c}$$

The time t_c' it takes the back of the pole to move from door A to door B is also the time the vaulter remains in the barn from his own point of view.

d) Special relativity tells us events which are simultaneous in the farmer's frame are not necessarily simultaneous in the vaulter's frame. Indeed, in the vaulter's frame door B closes and opens well before door A and no wood splinters.