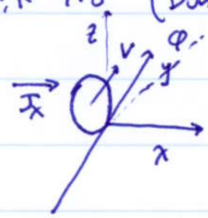
Solutions to MT2 2007

1) K;K 7.8 (Done in precept!) In the limit that the hoop



acts like a gyroscope, a torque will cause it to precess. $\vec{L} = I\omega = \frac{IV}{R} = \frac{mR^2V}{R} = -mRV\,\hat{x}$

$|\Delta\vec{L}| = \int \tau\,dt = RJ_x$ in the direction $+\hat{y}$. Thus the hoops makes an angle of $\frac{\Delta L}{L} = \frac{RJ_x}{mRV} = \frac{J_x}{mV}$ with respect to the y axis as shown.

2) a) The shape of the orbit is determined by the eccentricity and/or energy. $E_i = -\frac{GMm}{2R}$, $\ell_i^2 = GMm^2R$, $\mu = m$.

Angular momentum is conserved and so after the "blow up" $E_f = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{G3Mm}{4r}$. At $\dot{r} = 0 = -\frac{GMm}{4R}$

Thus the orbit is an ellipse. Note that if 1/2 the mass of the star blew away $E_f = \frac{\ell^2}{2mR^2} - \frac{GMm}{2R} = \frac{GMm^2R}{2mR^2} - \frac{GMm}{2R} = 0$ and the orbit would be parabolic.

b) Many ways to solve. Here is one

$$r_0 \equiv \frac{\ell^2}{\mu C} = \frac{GMm^2R}{m\frac{3}{4}GMm} = \frac{4}{3}R$$

$$\varepsilon = \left(1 + \frac{2E\ell^2}{\mu C^2}\right)^{1/2} = \left(1 - \frac{2\,GMm\,GMm^2R}{4Rm\,G^2\left(\frac{3}{4}M\right)^2m^2}\right)^{1/2} = \frac{1}{3}$$ consistent with the above

$$r_{max} = \frac{r_0}{1-\varepsilon} = \frac{\frac{4}{3}R}{1-\frac{1}{3}} = 2R$$

$$r_{min} = \frac{r_0}{1+\varepsilon} = \frac{\frac{4}{3}R}{1+\frac{1}{3}} = R$$

Problem 4(a): We want to find the solution of the equation

$$\ddot{y} + 2\omega_0\dot{y} + \omega_0^2 y = A_0\omega_0^2 cos(\omega t). \tag{1}$$

Recall that $e^{i\omega t} = cos(\omega t) + isin(\omega t)$. Introduce the companion equation

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = A_0\omega_0^2 sin(\omega t). \tag{2}$$

Multiplying Eq. (2) by i and adding to Eq. (1) yields

$$\ddot{z} + 2\omega_0\dot{z} + \omega_0^2 z = A_0\omega_0^2 e^{i\omega t}. \tag{3}$$

Try a solution of the form $z = Be^{i\omega t}$, where B is a complex number. This results in $\dot{z} = i\omega Be^{i\omega t}$ and $\ddot{z} = -\omega^2 Be^{i\omega t}$. Upon substituting these into Eq. (3) we obtain $B(\omega_0^2 - \omega^2 + 2i\omega_0\omega) = A_0\omega_0^2$. After rearranging terms we obtain

$$B = A_0\omega_0^2/(\omega_0^2 - \omega^2 + 2i\omega_0\omega). \tag{4}$$

In polar form, $B = Ae^{i\phi}$, where $A = \sqrt{BB^*}$ and $\phi = ArcTan(Im[B]/Re[B])$. We would like to solve for the amplitude, A, and the phase, ϕ.

It is helpful to rewrite B in Cartesian form (e.g. $B = u + iv$). Multiply the right hand side of Eq. (4) by 1 and simplify.

$$B = \frac{A_0\omega_0^2}{\omega_0^2 - \omega^2 + 2i\omega_0\omega} * \frac{\omega_0^2 - \omega^2 - 2i\omega_0\omega}{\omega_0^2 - \omega^2 - 2i\omega_0\omega} = \frac{A_0\omega_0^2(\omega_0^2 - \omega^2 - 2i\omega_0\omega)}{(\omega_0^2 - \omega^2)^2 + (2\omega_0\omega)^2}. \tag{5}$$

Now we can solve for A

$$A = \sqrt{BB*} = \sqrt{\frac{A_0\omega_0^2(\omega_0^2 - \omega^2 - 2i\omega_0\omega)}{(\omega_0^2 - \omega^2)^2 + (2\omega_0\omega)^2} * \frac{A_0\omega_0^2(\omega_0^2 - \omega^2 + 2i\omega_0\omega)}{(\omega_0^2 - \omega^2)^2 + (2\omega_0\omega)^2}} = A_0\omega_0^2\sqrt{\frac{1}{(\omega_0^2 - \omega^2)^2 + (2\omega_0\omega)^2}}. \tag{6}$$

The solution for ϕ is

$$\phi = \arctan(\frac{Im[B]}{Re[B]}) = \arctan(\frac{2\omega_0\omega}{\omega^2 - \omega_0^2}). \tag{7}$$

The complete solution is z=A$e^{i\phi}e^{i\omega t}$. The real part of this is y=$A\cos(\omega t + \phi)$.

Problem 4(b,c):

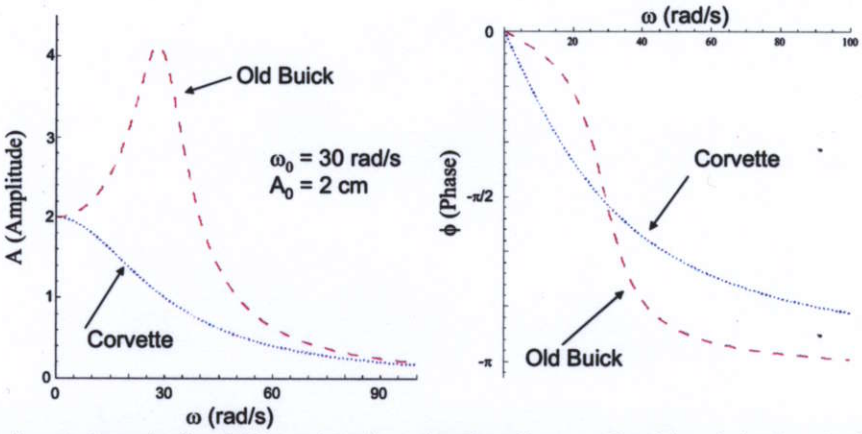


FIG. 1: Amplitude, A, and phase, ϕ, of oscillations plotted as a function of ω. ω_0=30 rad/s and A_0=2 cm in these plots. A=A_0 and ϕ=0 in the limit $\omega \longrightarrow 0$. On resonance ($\omega = \omega_0$), $A = A_0/2$ and ϕ=-$\pi/2$. In the limit $\omega \longrightarrow \infty$, $A = 0$ and ϕ=-π. The amplitude for the Corvette is maximum when $\omega = 0$. The frequency response of the "Old Buick" is shown by the dashed lines.

Problem 4(d): $y = A\cos(\omega t + \phi)$ so $\dot{y} = -A\omega\sin(\omega t + \phi)$. When $\phi = -\pi/2$, $\dot{y} = A\omega\cos(\omega t)$. We know from part (b) that $\phi = -\pi/2$ when $\omega = \omega_0$. Our task is to find the speed for which the car experiences a driving frequency of 1 Hz. The bumps are spaced 1 meter apart, therefore the car will experience a driving frequency of 1 Hz when it travels at a speed of 1 m/s=2.25 miles/hr. From parts (a,b), the amplitude of the oscillations is equal to $A_0/2$=1 cm on resonance ($\omega = \omega_0$).