PHY301 Final Solutions Fall 2008

Problem 1 PS #4 pb 4.7 Final on Jan 15, 2009

a) $$Z = \sum_{\text{states}} e^{-\epsilon/\tau}$$

$$= \sum_{\{s_1, s_2, s_3, \ldots\}} e^{-\sum_i s_i \hbar\omega_i/\tau}$$

$$= \left(\sum_{s_1=0}^{\infty} e^{-s_1\hbar\omega_1/\tau}\right)\left(\sum_{s_2=0}^{\infty} e^{-s_2\hbar\omega_2/\tau}\right)\cdots$$

$$= \prod_i \sum_{s_i=0}^{\infty} e^{-s_i\hbar\omega_i/\tau}$$

$$\boxed{Z = \prod_{(\text{modes})\, i} \frac{1}{1 - e^{-\hbar\omega_i/\tau}}}$$ 10 pts.

b) $$F = -\tau \ln Z$$

$$= +\tau \sum_{\text{modes } i} \ln\left\{1 - e^{-\hbar\omega_i/\tau}\right\}$$

$$= \underset{\substack{\uparrow \\ \text{polarization}}}{2\times}\ \tau \frac{V}{(2\pi)^3} 4\pi \int_0^{\infty} k^2 dk \ln\left\{1 - e^{-\hbar ck/\tau}\right\}$$

$$= 2\times\ \tau \frac{V}{2\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 \int_0^{\infty} x^2 dx \left\{1 - e^{-x}\right\}$$

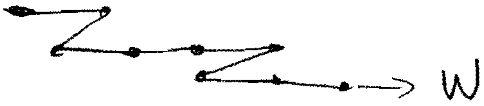
$$\boxed{F = -\frac{\tau^4 V \pi^2}{45\, \hbar^3 c^3}}$$ 10

c) $$\boxed{\sigma = -\frac{\partial F}{\partial \tau}\Big|_V = \frac{4\pi^2\tau^3 V}{45\, \hbar^3 c^3}}$$ 10

d) $\boxed{\mu = 0}$ 5 because photon 5 number is not conserved (can created, destroyed)

From midterm but with grand canonical ensemble

② 1D rubber band in the grand canonical ensemble.



$\overleftarrow{Wa} \rightarrow -Wa$

a) $$Z_{Grand} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_1^{N} = \frac{1}{1-e^{\beta\mu}Z_1}$$

$$Z_1 = e^{-Wa\beta} + e^{Wa\beta} = 2\cosh Wa\beta .$$

$$Z_{Grand} = \frac{1}{1-2e^{\beta\mu}\cosh Wa\beta}$$

b) μ? given $\langle N\rangle = N_0$

$$N_0 = \langle N\rangle = \tau\frac{\partial}{\partial\mu}\left(\ln Z_{Grand}\right)$$

$$= \tau(-)\frac{\partial}{\partial\mu}\ln\left(1-2e^{\beta\mu}\cosh Wa\beta\right)$$

$$= \cancel{-\tau}\,\frac{\cancel{-\beta}\,2e^{\beta\mu}\cosh Wa\beta}{1-2e^{\beta\mu}\cosh Wa\beta}$$

$$N_0\left(1-2e^{\beta\mu}\cosh Wa\beta\right) = 2e^{\beta\mu}\cosh Wa\beta .$$

$$\Rightarrow e^{\beta\mu}\cosh Wa\beta \cdot 2(1+N_0) = N_0$$

$$\Rightarrow \mu = \tau\ln\left[\frac{N}{(1+N_0)2\cosh Wa\beta}\right]$$

Problem 3:

a) $\mathcal{D}(\epsilon) = ?$

$\epsilon(k) = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \sqrt{\frac{2m\epsilon}{\hbar^2}}, \quad dk = \sqrt{\frac{2m}{\hbar^2}} \cdot \frac{1}{2\sqrt{\epsilon}} d\epsilon$

$\mathcal{D}(k)dk = \frac{A}{(2\pi)^2} \cdot 2\pi k \, dk \times 2_{spin}$

$= \frac{2A}{2\pi} k \, dk.$

$= 2 \times \frac{A}{2\pi} \frac{2m}{\hbar^2} \sqrt{\epsilon} \frac{d\epsilon}{2\sqrt{\epsilon}}$

$\boxed{\mathcal{D}(\epsilon)d\epsilon = \left(\frac{Am}{\pi\hbar^2}\right) d\epsilon}$ 10

b) $N = \int_0^{\epsilon_F} d\epsilon \, \mathcal{D}(\epsilon)$ using $f(\epsilon) \underset{T\to 0}{=} \Theta(\epsilon_F - \epsilon)$

$= \left(\frac{Am}{\pi\hbar^2}\right) \epsilon_F$

$\Rightarrow \boxed{\epsilon_F = \left[\frac{\pi\hbar^2}{m}\right]\left(\frac{N}{A}\right)}$ 10

c) At finite T,

$N = \int_0^{\infty} d\epsilon \, f(\epsilon) \, \mathcal{D}(\epsilon) \, d\epsilon.$

$$= \frac{Am}{\pi\hbar^2}\int_0^\infty d\epsilon \; \frac{1}{e^{(\epsilon-\mu)/\tau}+1}$$

$$= \frac{Am}{\pi\hbar^2}\int_0^\infty d\epsilon \; \frac{1}{1 + \left(e^{-\mu/\tau}\right) e^{\epsilon/\tau}}$$

Use formula given w/ $a = e^{-\mu/\tau}$, $b = 1/\tau$, $x = \epsilon$

$$\boxed{\cancel{N = \frac{Am}{\pi\hbar^2}\int_0^\infty}}$$

$$\frac{\pi\hbar^2}{m}\frac{N}{A} = \epsilon - \tau \ln\left\{1 + e^{\frac{\epsilon-\mu}{\tau}}\right\}\Bigg|_0^\infty$$

$\{\epsilon \to \infty$ ignore 1 in $\ln(1+e^{(\epsilon-\mu)/\tau})\}$

$$= \left\{\epsilon - \tau\,\frac{(\epsilon-\mu)}{\tau}\right\} - \left\{-\tau \ln\left[1 + e^{-\mu/\tau}\right]\right\}$$

$$= \mu + \tau \ln\left\{1 + e^{-\mu/\tau}\right\}.$$

$$\epsilon_F = \mu + \tau \ln\left\{\frac{e^{\mu/\tau}+1}{e^{\mu/\tau}}\right\}.$$

$$= \tau \ln\left\{e^{\mu/\tau} + 1\right)$$

$$\Rightarrow e^{\mu/\tau} + 1 = e^{\epsilon_F/\tau}$$

10

$$\Rightarrow \boxed{\mu = \tau \ln\left[e^{\epsilon_F/\tau} - 1\right]}$$

Problem (4) For both spin waves and phonons, use

$$\langle n(\omega)\rangle = \frac{1}{e^{\hbar\omega/\tau}-1} \quad \text{(Planck distrib. fn)}$$

$$U = \sum_{\text{modes}} (\text{energy of mode}) \times (\text{avg. occupancy of mode})$$

$$= \sum_{\omega} \hbar\omega \cdot \frac{1}{e^{\hbar\omega/\tau}-1}$$

(Told to assume) ∞

$$= \frac{V}{(2\pi)^3} 4\pi \int_0^{\infty} dk\, k^2 \frac{\hbar\omega(k)}{e^{\hbar\omega(k)/\tau}-1}$$

$$= \frac{V}{2\pi^2} \int_0^{\infty} dk\, k^2 \frac{\hbar\omega(k)}{e^{\hbar\omega(k)/\tau}-1}$$

i) Spin waves $\omega(k) = Jk^2$

So $x = \frac{\hbar\omega(k)}{\tau} = \frac{\hbar J k^2}{\tau}$.

$$\Rightarrow k = \left(\frac{\tau}{\hbar J}\right)^{1/2} x^{1/2}$$

$$dk = \left(\frac{\tau}{\hbar J}\right)^{1/2} \frac{1}{2} x^{-1/2}\, dx.$$

$$\Rightarrow U_{sw} = \frac{V}{2\pi^2} \int_0^{\infty} \frac{dx}{2\sqrt{x}} \left(\frac{\tau}{\hbar J}\right)^{3/2} x^{1/2} \cdot \frac{\tau x}{e^{x}-1}.$$

$$= J^{-3/2} \tau^{5/2} \underbrace{\left(\frac{V}{4\pi^2}\right)\hbar^{-3/2}}_{=A} \underbrace{\int_0^{\infty} dx \frac{x^{3/2}}{e^{x}-1}}_{=I_{sw}}$$

$$= A I_{sw} J^{-3/2} \tau^{5/2}$$

$$\gamma = -3/2, \quad \alpha = 5/2$$

ii) Phonons.

$$\omega(k) = c_s k.$$

$$x = \frac{\hbar c_s k}{\tau}; \quad dx = \frac{\hbar c_s dk}{\tau}.$$

$$\Rightarrow U_{ph} = \frac{V}{2\pi^2} \int_0^\infty dx \left(\frac{\tau}{\hbar c_s}\right)^3 x^2 \tau \, \frac{x}{e^x - 1}$$

5

$$= \underbrace{\frac{V}{2\pi^2 (\hbar c_s)^3}}_{B} \tau^4 \underbrace{\int_0^\infty dx \frac{x^3}{e^x - 1}}_{I_{ph}}$$

$$= B I_{ph} \tau^4 \qquad (\beta = 4)$$

b) $$C_{ph} = 4 B I_{ph} \tau^3$$

$$C_{sw} = \frac{5}{2} A J^{-3/2} I_{sw} \tau^{3/2}$$

$$\frac{A}{B} = \frac{\frac{V}{4\pi^2} \times \hbar^{3/2}}{\frac{V}{2\pi^2 (\hbar c_s)^3}}$$

$$C_{sw}(\tau_e) = C_{ph}(\tau_e)$$

10

$$\Rightarrow 4 B I_{ph} \tau_e^3 = \frac{5}{2} A J^{-3/2} I_{sw} \tau_e^{3/2}$$

$$\Rightarrow \tau_E^{3/2} = \frac{5}{8} \frac{A I_{sw} J^{-3/2}}{B I_{ph}}.$$

$$\tau_E = \left(\frac{5}{8} \frac{A I_s}{B I_{ph}}\right)^{2/3} J^{-1}; \quad \left[J = \left(\frac{5 A I_s}{8 B I_{ph}}\right)^{2/3} \right.$$

Problem 5   See Class notes L27 & K&K 10.6 on PS9

there are a number of ways to solve this problem. Following the derivation of the Clausius-Clapeyron equation

$$dG_N = dG_S \quad \text{along the transition/phase boundary}$$

$$-\sigma_N d\tau - \mu_0 M_N dH = -\sigma_S d\tau - \mu_0 M_S dH$$

For a superconductor $M_N = 0$

$$M_S = -VH_c$$

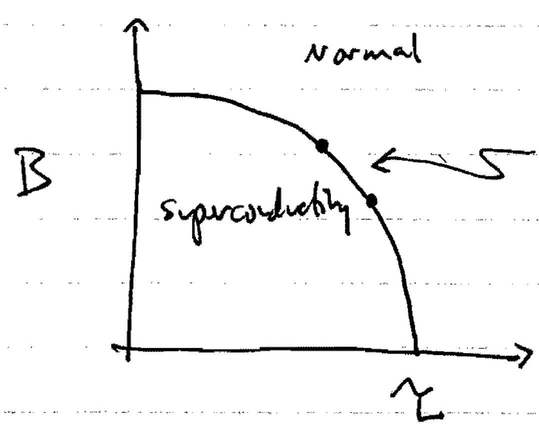
$$\Rightarrow (-\sigma_N + \sigma_S) d\tau = +\mu_0 V H_c \, dH_c \quad \text{along boundary}$$

$$\Rightarrow \frac{\sigma_S - \sigma_N}{V} = \mu_0 H_c \frac{dH_c}{d\tau}$$

$$\frac{\sigma_S - \sigma_N}{V} = \frac{1}{\mu_0} B_c \frac{dB_c}{d\tau}$$

The latent heat per volume $= l = \frac{(\sigma_N - \sigma_S)\tau}{V}$

l is positive $\sigma_N > \sigma_S$



$$l = -\frac{\tau}{\mu_0} B_c \frac{dB_c}{d\tau}$$

slope is negative $\Rightarrow$ l is positive

(6) Joule Thompson.

start P_L and V_L, finish P_R and V_R.

a) Heat exchange is 0. ($\Delta Q = 0$)

$\Rightarrow U_f - U_i$ = work done <u>on</u> gas.

$$U_R - U_L = P_L V_L - P_R V_R$$

$$U_R + P_R V_R = U_L + P_L V_L$$

$\Rightarrow H = \text{const}.$

b) gas is ideal $\Rightarrow H = \frac{5}{2} N\tau \propto \tau$.

$U = \frac{3}{2} N\tau \propto \tau$.

If $H = \text{const}$ then $\tau = \text{const}$ then $U = \text{const}$.

So $U_f - U_i = 0$.

<u>Note</u>: writing $U_f - U_i = \frac{3}{2} N(\tau_f - \tau_i) = \frac{3}{2} N(P_R V_R - P_L V_L)$,
or writing $U_f - U_i = P_L V_L - P_R V_R$ would get most of the credit, since P_L, V_L, P_R, V_R are given in the problem.
(although $\Delta U = -\frac{3}{2} \Delta U$ should make you realize $\Delta U = 0$)

c) $$F = -N\tau\left(\ln \frac{n_Q(V-Nb)}{N} + 1\right) - \frac{N^2 a}{V}$$

$$\sigma = -\frac{\partial F}{\partial \tau} = N\left(\ln \frac{n_Q(V-Nb)}{N} + 1\right) + N\tau\left(\frac{\cancel{N}}{\cancel{n_Q(V-Nb)}} \frac{\cancel{(V-Nb)}}{\cancel{N}} \frac{\partial n_Q}{\partial \tau}\right)$$

$\uparrow$ $\frac{3}{2} \frac{\cancel{n_Q}}{\cancel{\tau}}$ since $n_Q \propto \tau^{3/2}$

$$\tau\sigma = N\tau\left(\ln \frac{n_Q(V-Nb)}{N} + 1\right) + \frac{3}{2} N\tau$$

$$U = F + \tau\sigma = \frac{3}{2} N\tau - \frac{N^2 a}{V}.$$

d) want $\tau_2 < \tau_1$ if $V_2 > V_1$.

Put $V_2 = V_1 + \Delta$.

Demand $H_2 = H_1$.

$$PV = \frac{N\tau V}{V - Nb} - \frac{N^2 a}{V} \qquad \text{from vdw eq.}$$

$$H = U + PV$$

$$H = \frac{3}{2}N\tau - \frac{2N^2 a}{V} + \frac{N\tau V}{V - Nb}.$$

$$H_1 = H_2$$

$$\Rightarrow \frac{3}{2}N\tau_1 - \cancel{\frac{2N^2 a}{V_1}} + \cancel{\frac{N\tau_1 V_1}{V_1 - Nb}} = \frac{3}{2}N\tau_2 - \frac{2N^2 a}{V_1 + \Delta} + \frac{N\tau_2(V_1+\Delta)}{V_1 - Nb + \Delta}$$

$$= \frac{3}{2}N\tau_2 - \cancel{\frac{2N^2 a}{V_1}} + \frac{2N^2 a\Delta}{V_1^2} + \cancel{\frac{N\tau_2 V_1}{V_1 - Nb}} + \frac{N\tau_2\Delta}{V_1 - Nb} - \frac{N\tau_2 V_1\Delta}{(V_1 - Nb)^2}$$

$$0 < \frac{3}{2}N(\tau_1 - \tau_2) = \frac{2N^2 a\Delta}{V_1^2} + \cancel{\frac{N\tau_2\Delta}{V_1}} + \frac{N^2 b\tau_2\Delta \cancel{V_1}}{V_1^{\cancel{3}}} - \cancel{\frac{N\tau_2 V_1\Delta}{V_1^2}} - \frac{N^2\tau_2 \cancel{V_1}\Delta 2b}{V_1^{\cancel{3}}}$$

expand above in small b.

$$= \frac{2N^2 a\Delta}{V_1^2} - \frac{N^2 b\tau_2\Delta}{V_1^2}$$

$$0 < \frac{2\cancel{N^2} a\cancel{\Delta}}{\cancel{V_1^2}} - \frac{\cancel{N^2} b\tau_2\cancel{\Delta}}{\cancel{V_1^2}}$$

$$0 < 2a - b\tau_2$$

$$b\tau_2 < 2a.$$

Problem 7 See Class notes L35 ; K&K pg 410 & 14.5 PS#11

In steady state the Boltzmann equation becomes

$$\vec{v}\cdot\vec{\nabla}_r f = -\frac{f-f_0}{\tau_c}$$ in the relaxation approximation.

For flow in the x direction

$$v_x \frac{\partial f}{\partial x} = -\frac{f-f_0}{\tau_c}$$

Now $\frac{\partial f}{\partial x} \sim \frac{\partial f_0}{\partial x} = \frac{\partial n}{\partial x}\frac{f_0}{n}$

$$\Rightarrow v_x \frac{\partial n}{\partial x}\frac{f_0}{n} = -\frac{f}{\tau_c} + \frac{f_0}{\tau_c} \quad \text{or} \quad f = f_0 - \tau_c v_x \frac{\partial n}{\partial x}\frac{f_0}{n}$$

$$J_x = \underbrace{\int v_x f_0 \, d^3v}_{\text{Odd fcn of } v \Rightarrow 0} - \int \tau_c v_x^2 \frac{\partial n}{\partial x}\frac{f_0}{n} d^3v = -D\frac{\partial n}{\partial x}$$

$$\Rightarrow D = \tau_c \int \frac{f_0}{n} v_x^2 \, d^3v \qquad f_0 = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}$$

There are a number of ways to do the integral.

$$① \quad = \tau_c\left(\frac{m}{2\pi kT}\right)^{3/2}\int_{-\infty}^{\infty} v_x^2 e^{-mv_x^2/2kT} dv_x \left[\int_{-\infty}^{\infty} e^{-mv_y^2/2kT} dv_y\right]^2$$

Recall $\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{1}{a}\Gamma(\tfrac{1}{2}) = \frac{\pi^{1/2}}{a}$ & $\int_{-\infty}^{\infty} x^2 e^{-a^2x^2} dx = \frac{\pi^{1/2}}{2a^3}$

$$= \tau_c \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2\left(\frac{m}{2kT}\right)^{3/2}} \frac{\pi^{1/2}}{1} \frac{\pi}{\frac{m}{2kT}}$$

$$= \frac{\tau_c kT}{m}$$

(2) $\varepsilon = \frac{1}{2} m v^2 \Rightarrow v_x^2 = \frac{2}{3}\frac{\varepsilon}{m} \qquad d^3v = 4\pi v^2 dv = 4\pi \left(\frac{2\varepsilon}{m}\right)^{1/2} \frac{d\varepsilon}{m}$

$$\Rightarrow D = \tau_c \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty \frac{2}{3}\frac{\varepsilon}{m} e^{-\varepsilon/\tau} 4\pi \left(\frac{2\varepsilon}{m}\right)^{1/2} \frac{d\varepsilon}{m}$$

$$= \frac{\tau_c}{m} \frac{1}{(kT)^{3/2}} \frac{1}{(2\pi)^{3/2}} \frac{2}{3} 4\pi\sqrt{2} \underbrace{\int_0^\infty \varepsilon^{3/2} e^{-\varepsilon/\tau} d\varepsilon}_{\tau^{5/2}\Gamma(5/2)}$$

$$= \frac{\tau_c kT}{m} \frac{1}{\pi^{1/2}} \frac{4}{3} \Gamma(5/2) \qquad \begin{aligned} \Gamma(1/2) &= \pi^{1/2} \\ \Gamma(3/2) &= \pi^{1/2}/2 \\ \Gamma(5/2) &= 3\pi^{1/2}/4 \end{aligned}$$

(3) Note that

$$v_{rms}^2 = \frac{\int v^2 f_0 \, d^3v}{\int f_0 \, d^3v} = n^{-1}\int v^2 f_0 \, d^3v = 3n^{-1}\int v_x^2 f_0 \, d^3v = \frac{3kT}{m}$$

$$\Rightarrow$$

$$D_c = \frac{\tau_c}{n}\int f_0 v_x^2 d^3v = \frac{\tau_c}{n}\frac{nkT}{m} = \frac{\tau_c kT}{m}$$

Note that we did not put much weight on doing the integral as long as the units were correct.