PHY 301 **PROBLEM SET 2 (SOLUTIONS)**

① (Reif 2.5)

We have an infinitesimal quantity

$$đF \equiv A\,dx + B\,dy$$

where A and B are both fns. of x and y.

a) We suppose $đF$ is an exact differential so that $F = F(x,y)$. In this case, note that

$$\begin{aligned} dF &= F(x+dx, y+dy) - F(x,y) \\ &= F(x+dx, y+dy) - F(x,y+dy) + F(x,y+dy) - F(x,y) \\ &= \underbrace{\frac{\partial F}{\partial x}}\,dx + \frac{\partial F}{\partial y}\,dy \end{aligned}$$

note that this is technically at ~~the~~ the pt. $(x, y+dy)$, but the correction is small (vanishing as dy is infinitesimal)

This can be compared with $dF \equiv đF = A(x,y)\,dx + B(x,y)\,dy$ and so

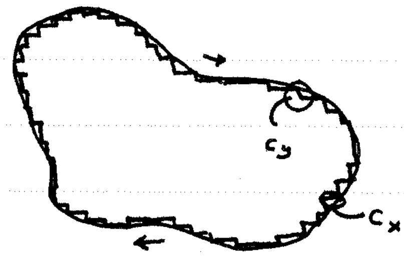
$$A(x,y) = \frac{\partial F}{\partial x}, \quad B(x,y) = \frac{\partial F}{\partial y}$$

$$\Rightarrow \quad \frac{\partial A}{\partial y} = \frac{\partial^2 F}{\partial y\,\partial x} = \frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial B}{\partial x}.$$

b) If dF is an exact differential, we wish to show that the integral $\oint đF$ evaluated along any closed path in the xy plane must vanish.

~~[illegible]~~ ~~gimme~~ Here's a non-rigorous but essentially correct proof:

Consider a closed path in the xy plane, C



Break the curve C up into infinitesimal segments C_{x_i} $\parallel^{l}$ to the x-axis, C_{y_i} parallel to the y-axis

(Each $C_{x_i} = (x_i, y_i) \to (x_{i+1}, y_i)$
$C_{y_i} = (x_{i+1}, y_i) \to (x_{i+1}, y_{i+1})$.)

$$\oint_C dF = \oint_C (A\,dx + B\,dy)$$

$$= \sum_i \left(\int_{C_{x_i}} A\,dx + \int_{C_{y_i}} B\,dy \right)$$

since $dy \to 0$ on each C_{x_i}, $dx \to 0$ on each C_{y_i}

$$= \sum_i \int_{C_{x_i}} \frac{\partial F}{\partial x} dx + \int_{C_{y_i}} \frac{\partial F}{\partial y} dy.$$

$$= \sum_i \{F(x_{i+1}, y_i) - F(x_i, y_i)\} + \sum_i \{F(x_{i+1}, y_{i+1}) - F(x_{i+1}, y_i)\}.$$

~~Since~~ You should be able to convince yourself that the telescoping sum vanishes. So

$$\boxed{\oint_C dF = 0}$$ as required

[Note that we use $x_{i+1} \equiv x_i + \delta x_i$, etc. where we let $\delta x_i \to 0$. at ~~are~~ the end of the argument.] ∎

② (Reif 2.7)

a) We have a particle confined in a cube of dimension $L_x \times L_y \times L_z$. The energy is given by

$$E = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

If we change the length quasistatically, following eqns (2.9.1) – (2.9.4), the work done by the system is

$$dW = -\left(\frac{\partial E}{\partial L_x}dx + \frac{\partial E}{\partial L_y}dy + \frac{\partial E}{\partial L_z}dz\right) = -\frac{\partial E}{\partial \vec{L}} \cdot d\vec{x}.$$

But since the infinitesimal amount of work is

$$dW = \vec{F} \cdot d\vec{x}$$

which gives $\quad F_i = \dfrac{-\partial E}{\partial L_i} \quad i = x, y, z.$

So in particular, the force on a wall perpendicular to the x-axis is $\quad F_x = \dfrac{-\partial E}{\partial L_x} = +\dfrac{\hbar^2 \pi^2 n_x^2}{m L_x^3}$

b) The force per unit area on this wall is

$$p_x = \frac{F_x}{L_y L_z} = \frac{+\hbar^2 \pi^2 n_x^2}{m L_x^3 L_y L_z} = +\frac{\hbar^2 \pi^2}{mV}\frac{n_x^2}{L_x^2}$$

Now $\quad \langle p_x \rangle = \dfrac{+\hbar^2 \pi^2}{mV}\dfrac{\langle n_x^2 \rangle}{L_x^2} \quad$ ($\langle \ldots \rangle$ is alternative notation for $\overline{(\ldots)}$)

Also, must have $\quad \langle n_x^2 \rangle = \langle n_y^2 \rangle = \langle n_z^2 \rangle \equiv n_0^2$

So $\quad \langle n_x^2 \rangle = \dfrac{1}{3}\left(\langle n_x^2 \rangle + \langle n_y^2 \rangle + \langle n_z^2 \rangle\right).$

$$\langle E \rangle = \frac{\hbar^2 \pi^2}{2m}\left(\frac{\langle n_x^2 \rangle}{L_x^2} + \frac{\langle n_y^2 \rangle}{L_y^2} + \frac{\langle n_z^2 \rangle}{L_z^2}\right)$$

$$= \frac{\hbar^2 \pi^2}{2m} n_0^2 \left(\frac{1}{L_x^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}\right)$$

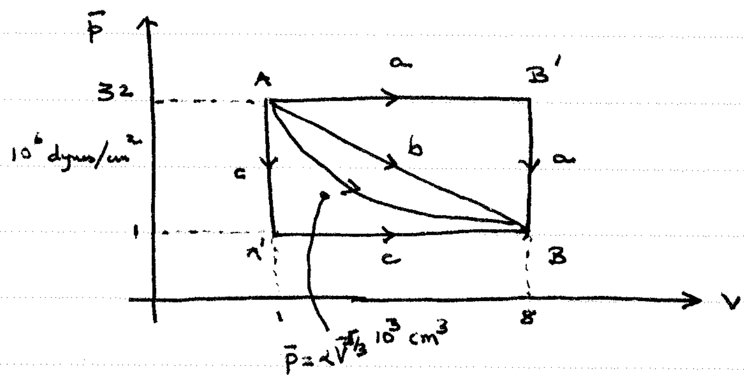
Since $L_x = L_y = L_z$, $\langle E \rangle = 3\,\frac{\hbar^2}{2m}\,\frac{n_0^2}{L_x^2}$

Therefore $p_x = \frac{2}{V}\overbrace{\left(\frac{\hbar^2\pi^2}{2m}\,\frac{n_0^2}{L_x^2}\right)}^{=\langle E\rangle/3} = \frac{2\langle E\rangle}{3V}$. ■

③
(Reif 2.11)

We have that in a quasistatic process $A \rightarrow B$ in which no heat is exchanged with the environment, the mean pressure $\bar{p}$ of a certain amount of gas is found to change with its volume V as

$$\bar{p} = \alpha V^{-5/3}$$



a) The system is expanded from original to final volume, while heat is added to maintain pressure constant; then volume is kept constant and heat is extracted to reduce the pressure to 10^6 dynes/cm².

Since the pressure is constant, $dW = \bar{p}\,dV$, so the total work done is

$$W = \int_{V_A}^{V_{B'}} \bar{p}\, dV = \bar{p}\,(V_{B'} - V_A) = 32\times10^6\ \frac{\text{dynes}}{\text{cm}^2} \times 7 \times 10^3\ \text{cm}^3$$

$$= 22.4\times10^9\ \text{erg}$$

$$= 22{,}400\ \text{J}.$$

Along the path AB (in which $\bar{p} = \alpha V^{-5/3}$), $đQ = 0$.
So $d\bar{E} = -đW$ along this path.

$$\Delta\bar{E} = -\int_{V_A}^{V_B} \bar{p}\, dV = +\frac{3}{2}\alpha\left\{V_B^{-2/3} - V_A^{-2/3}\right\}$$

Using either A or B, $\alpha = 32 \times 10^{11}$ dynes cm^3. and so we find

$$\Delta\bar{E} = \frac{3}{2} \cdot 32 \times 10^{11} \cdot \left[\frac{1}{4} - 1\right] \cdot 10^{-2} \text{ erg} = -3600\text{J}.$$

Since $\bar{E}$ is a state variable, the energy change along AB is
$\Delta E_{BA} = E_B - E_A = (E_B - E_{B'}) + (E_{B'} - E_A) = \Delta E_{BB'A}$. Then we have $\Delta Q = \Delta\bar{E} + \Delta W = \underline{18,800\text{J}}$.

b) The volume is increased and heat is supplied to cause the pressure to decrease linearly with the volume.

Along the line (b), have $p(V) = \frac{255}{7} - \frac{31}{7}V$.

$$\text{So } \Delta W = \int_{V_A}^{V_B} p\,dV = \left[\frac{255}{7}(8-1) - \frac{31}{7}\left(\frac{8^2 - 1^2}{2}\right)\right] \times 10^9 \text{ erg}.$$

Work done is $\Delta W = \underline{11550 \text{ J}}$ $\quad(10^7 \text{ erg} = 1\text{J})$
Heat change is
$\Rightarrow \Delta Q = \Delta E + \Delta W = 7950 \text{ J}$
(we used ΔE from above, again since E is a state variable)

c) If we perform the steps of (a) in opposite order, the change in internal energy ΔE is not affected as E is a state variable. However, the gas expands at a lower pressure, so that the work done is

$$\Delta W = \int_{V_{A'}}^{V_B} \bar{p}\, dV = 1 \times 10^6 \text{ dyne/cm}^2 \cdot 7 \times 10^3 \text{ cm}^3 = 700 \text{ J}.$$

The heat change is $\Delta Q = \Delta E + \Delta W = (-3600 + 700)\text{ J} = \underline{-2900\text{ J}}$

(4) Reif.

We have a box with a partition that divides the volume in a 3:1 ratio. The larger portion contains 1000 molecules of Ne, smaller 100 molecules of He; a hole is punctured and equilibrium is attained.

a) A given gas molecule will be on the bigger side w/ probability 3/4, on the smaller w/ probability 1/4. Thus

$$\overline{N}_{Ne,\,bigger} = 750, \qquad \overline{N}_{Ne,\,smaller} = 250$$

$$\overline{N}_{He,\,bigger} = 75, \qquad \overline{N}_{He,\,smaller} = 25$$

b) The probability that all Ne molecules are still in the larger portion is $(3/4)^{1000}$ independent of the positions of the He molecules; similarly the prob. that all the He molecules are in the smaller portion is $(1/4)^{100}$ independent of the positions of the Ne molecules. Thus, the prob. that both these independent events occur is $\left(\frac{1}{4}\right)^{100}\left(\frac{3}{4}\right)^{1000}$.

Problem 5

Reif 3.3

the Gaussian approx (see P.S. 1, #7)

$$\Omega(E) = \sqrt{\frac{2}{N\pi}}\, 2^N e^{-\frac{2}{N}\left(\frac{E}{2\mu H}\right)^2}$$

$$\Omega(E') = \sqrt{\frac{2}{N'\pi}}\, 2^{N'} e^{-\frac{2}{N'}\left(\frac{E'}{2\mu' H}\right)^2}$$

a) When the two systems are in thermal equilibrium $\beta = \beta'$,

where $\beta = \dfrac{\partial \ln \Omega}{\partial E} = \dfrac{\partial}{\partial E}\left[\ln(\dots) - \frac{2}{N}\left(\frac{E}{2\mu H}\right)^2\right]$

$$\beta = -\frac{1}{N}\left(\frac{E}{\mu^2 H^2}\right)$$

so $\beta = \beta' \Rightarrow \boxed{\dfrac{\tilde{E}}{N\mu^2} = \dfrac{\tilde{E}'}{N'\mu'^2}}$

b) by conservation of energy $\tilde{E} + \tilde{E}' = bN\mu H + b'N'\mu' H$

plug in a) $\left(1 + \frac{N'}{N}\frac{\mu'^2}{\mu}\right)\tilde{E} = (bN\mu + b'N'\mu')H$

$$\boxed{\tilde{E} = HN\mu^2 \frac{bN\mu + b'N'\mu'}{N\mu^2 + N'\mu'^2}}$$

c) since $W = 0$, the heat flow into the system is equal to the change in energy of the system, which was originally $E_i = bN\mu H$

$$Q = HN\mu^2 \frac{bN\mu + b'N'\mu'}{N\mu^2 + N'\mu'^2}$$

$$\boxed{Q = HNN'\mu\mu' \frac{b'\mu - b\mu'}{N\mu^2 + N'\mu'^2}}$$

d) $P(E) \sim \Omega(E)\, \Omega'(E_T - E)$

prob the other system has the right energy for conservation to work.

prob system has energy E

and let's worry about prefactors later

$$P(E) \sim \exp\left[-\frac{2}{N}\left(\frac{E}{2\mu H}\right)^2\right] \exp\left[-\frac{2}{N'}\left(\frac{E_T - E}{2\mu' H}\right)^2\right]$$

the product of two Gaussians is a Gaussian.
We can expand out what's in the exponential

$$-\left(\frac{1}{2N\mu^2H^2} + \frac{1}{2N'\mu'^2H^2}\right)E^2 + (\text{something})E + \text{something else}$$

which we know must look like

$$-\frac{1}{2H^2}\left(\frac{\mu'^2N' + \mu^2 N}{\mu^2\mu'^2 NN'}\right)(E - \tilde{E})^2$$

because it has to be a Gaussian, centered on $\tilde{E}$

and we can read off $\frac{1}{2\sigma^2} = \frac{1}{2H^2}\left(\frac{\mu'^2N' + \mu^2 N}{\mu^2\mu'^2 NN'}\right)$

or $\boxed{\sigma = \sqrt{\frac{H^2\mu^2\mu'^2 NN'}{\mu'^2N' + \mu^2N}}}$

$$\boxed{P(E)\,dE = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(E-\tilde{E})^2/2\sigma^2}\, dE}$$

e)

f) $|\Delta E/E| = \sqrt{\frac{H^2\mu^2\mu'^2NN'}{\mu'^2N'+\mu^2N}} \Big/ HN\mu^2\frac{bN\mu + b'N'\mu'}{N\mu^2 + N'\mu'^2}$

$N' \gg N$
$b' \sim b,\ \mu' \sim \mu$

$$\sim \frac{\sqrt{N'}\mu}{N\mu^2 b'/\mu'} \boxed{= \frac{\mu'}{\sqrt{N'}b'\mu} \sim \frac{1}{\sqrt{N'}}}$$

Problem 6

Reif 3.4

The entropy change of the reservoir is $\Delta S' = -\dfrac{Q}{T'}$

The second law is $\Delta S + \Delta S' = \Delta S - \dfrac{Q}{T'} \geq 0$

or $\boxed{\Delta S \geq \dfrac{Q}{T'}}$

Problem 7

Reif 3.5 for an ideal gas, $\Omega \sim V^N$

a) for two species with total energy E_0

$$\Omega(E_1) = \Omega_1(E_1)\,\Omega_2(E_0 - E_1)\ \boxed{\sim V^{N_1+N_2}}$$

b) $\ln \Omega = (N_1 + N_2) \ln V + \ldots$

$$P = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \Omega = \frac{1}{\beta} \frac{N_1 + N_2}{V}$$

or $\boxed{\bar{p} V = (N_1 + N_2) kT}$