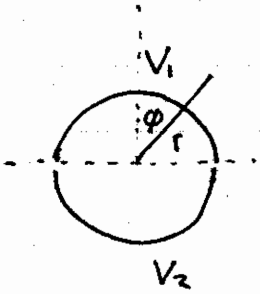
Solutions PH4304 May 2004

P1



V cannot diverge at the origin and so $b_0 = b_k = d_k = 0$

V must be symmetric about $\varphi = 0$ and so $c_k = d_0 = 0$

We'll relabel $a_k \to b_k$ to avoid confusion with the radius, a

Thus $V(r,\varphi) = b_0 + \sum_{k=1}^{\infty} b_k r^k \cos k\varphi$

at $r=0$ $V(r,\varphi) = \dfrac{V_1+V_2}{2} = b_0$ by symmetry

at $r=a$ $V(a,\varphi) = \sum_k b_k a^k \cos k\varphi$ need to find b_k

Use "Fourier's trick"

$$\int_0^{2\pi} V(a,\varphi)\cos k'\varphi \, d\varphi = \sum_k b_k a^k \underbrace{\int_0^{2\pi} \cos k\varphi \cos k'\varphi \, d\varphi}_{\pi\,\delta(k-k')}$$

$$\int_{-\pi/2}^{\pi/2} V_1 \cos k'\varphi \, d\varphi + \int_{\pi/2}^{3\pi/2} V_2 \cos k'\varphi \, d\varphi = b_{k'} a^{k'} \pi$$

$$V_1 \left.\frac{\sin k'\varphi}{k'}\right|_{-\pi/2}^{\pi/2} + V_2 \left.\frac{\sin k'\varphi}{k'}\right|_{\pi/2}^{3\pi/2} = b_{k'} a^{k'} \pi$$

$$V_1\, 2\,\frac{\sin k'\pi/2}{k'} + V_2\left[\frac{\sin k' 3\pi/2}{k'} - \frac{\sin k'\pi/2}{k}\right] = b_{k'} a^{k'} \pi$$

$$= \frac{2\sin k'\pi/2}{a^k \pi k'}(V_1 - V_2) = b_{k'}$$

$$\Rightarrow V(r,\varphi) = \frac{V_1+V_2}{2} + \frac{2(V_1-V_2)}{\pi}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{r}{a}\right)^k \sin\frac{k\pi}{2}\cos k\varphi$$

$$= \frac{V_1+V_2}{2} + \frac{(V_1-V_2)}{\pi}\tan^{-1}\left\{\frac{2ar\cos\varphi}{a^2-r^2}\right\}$$

$\boxed{P2}$ The potentials are

$$\begin{array}{lll} & r \le a & r \ge a \\ \vec{A} = & \frac{\mu_0 a \omega \sigma}{3} r \sin\theta\, \hat{\varphi} & \frac{\mu_0 a^4 \omega \sigma}{3} \frac{\sin\theta}{r^2} \hat{\varphi} \\ V = & \text{const} & \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \end{array}$$

$$\sigma = \frac{Q}{4\pi a^2}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

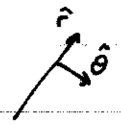
a) Inside $\quad \vec{E} = -\frac{\mu_0 a \sigma}{3} r \sin\theta \frac{d\omega}{dt} \hat{\varphi} \qquad \vec{B} = \frac{2}{3} \mu_0 \sigma a \omega$

Outside $\quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r} - \frac{\mu_0 a^4 \sigma \sin\theta}{3 r^2} \frac{d\omega}{dt} \hat{\varphi}$

$$\vec{B} = \frac{\mu_0 a^4 \omega \sigma}{3 r^3} \left[2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right] \quad \text{dipole}$$

b) Inside contribution averages to 0

Outside $\quad \vec{l}_{em} = \vec{r} \times \vec{P}_{em} = \vec{r} \times \varepsilon_0 (\vec{E} \times \vec{B})$



What terms contribute?

$\hat{r} \times \hat{\theta}$ in $\hat{\varphi}$ dir $\quad \hat{r} \times \hat{\varphi}$ in $-\hat{\theta}$ dir $\quad$ OK

$\hat{\varphi} \times \hat{r}$ in $\hat{\theta}$ $\quad \hat{r} \times \hat{\theta}$ in $\hat{\varphi}$ $\quad$ avg to 0.

$\hat{\varphi} \times \hat{\theta}$ in $-\hat{r}$ $\quad \hat{r} \times \hat{r} = 0$

$$\Rightarrow \vec{l}_{em} = -\frac{Q}{4\pi\varepsilon_0 r^2} \frac{\mu_0 a^4 \omega \sigma}{3 r^3} r \sin\theta\, \hat{\theta} = -\frac{Q \mu_0 a^4 \omega \sigma \sin\theta\, \hat{\theta}}{12\pi r^4}$$

$$\vec{L} = \int \vec{l}_{em}\, dV = \int_0^{\pi} \int_a^{\infty} \frac{Q \mu_0 a^4 \omega \sigma \sin^2\theta}{12\pi r^4} r^2 \sin\theta\, dr\, d\theta\, 2\pi\, \hat{z}$$

Only z survives by symmetry

$$= \frac{Q \mu_0 a^4}{6} \frac{\omega Q}{4\pi a^2} \int_0^{\pi} \int_a^{\infty} \frac{\sin^3\theta\, dr\, d\theta}{r^2} \hat{z} = \frac{Q^2 \mu_0 a \omega}{18\pi} \hat{z}$$

c) Angular momentum flowing out is $-\int (\vec r \times \overleftrightarrow{T})\cdot d\vec a$

$= -\int (\hat r \times \vec E)\, E_r\, da$ For E part. Notice that contribution for B will have the same form ∴ will avg to 0 so worry only about E part. at surface

$$\vec r \times \vec E = -a\,\frac{\mu_0 \cancel{a^2}\,\sigma \sin\theta}{3\cancel{a^2}}\,\frac{d\omega}{dt}\,(\hat r \times \hat\varphi)$$

$$\left| L \text{ flowing out} \right| = +\int \epsilon_0\,\frac{a\,\mu_0\,\sigma \sin\theta}{3}\,\frac{d\omega}{dt}\,2\pi a^2 \sin\theta\, d\theta\,\underbrace{\frac{Q}{4\pi\epsilon_0 a^2}}_{E_r}\,\underbrace{a\sin\theta}_{\hat z\,\text{comp}}$$

$$= \frac{\mu_0 a^{\cancel{5}}\sigma\,\epsilon_0\, d\omega}{3\,\cancel{a^2}\; dt}\,\frac{Q}{4\pi\epsilon_0 \cancel{a^2}}\,a^2\, 2\pi\,\underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{4\pi/3}\;\hat z$$

$$= \frac{\epsilon_0 \mu_0\, a\, Q^2}{3\cdot 4\cancel{\pi}\,\cancel{a^2}}\,\frac{d\omega}{dt}\,\frac{2\cancel{\pi}\cancel{a^2}\,4\cancel{\pi}}{4\cancel{\pi}\epsilon_0\;\cancel{3}} = \frac{\mu_0 Q^2 a}{18}\,\frac{d\omega}{dt}$$

$$= \frac{\mu_0 Q^2 a}{18}\,\frac{d\omega}{dt}$$

d) The angular momentum in the field outside the spinning shell "flows" from the shell

$$\frac{d}{dt}[p+L] = [p+c]$$

Torque

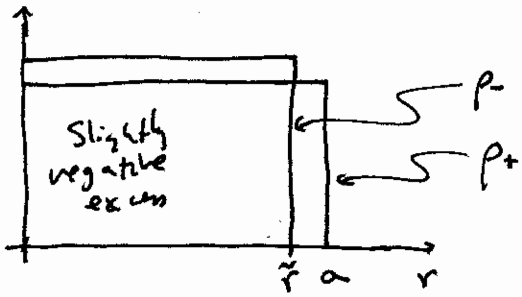
P3 a) Gauss's Law $$2\pi r l E_r = \frac{\rho_+ \pi r^2 l}{\varepsilon_0} + \int_0^r \frac{\rho_- 2\pi l r' dr'}{\varepsilon_0}$$

$$\text{or } E_r = \frac{\rho_+ r}{2\varepsilon_0} + \frac{1}{r\varepsilon_0}\int_0^r \rho_-(r') r' dr'$$

Ampere's law $$B_\varphi 2\pi r = \mu_0 \int_0^r v \rho_-(r') 2\pi r' dr'$$

$$\Rightarrow B_\varphi = \frac{\mu_0 v}{r}\int_0^r \rho_-(r') r' dr'$$

Now connect to the variables in the problem. We know that the negative charge bunch in



$$v\int_0^a \rho_+ 2\pi r' dr' = I \Rightarrow \rho_+ = \frac{I}{v\pi a^2}$$

By force balance with $\gamma^{-1} = (1 - v^2/c^2)^{1/2}$

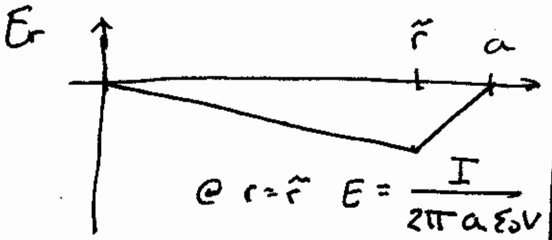
$$\frac{1}{\gamma^2}\int_0^{\tilde{r}} \rho_- r' dr' = -\frac{\rho_+ r^2}{2} \Rightarrow \boxed{\frac{\rho_-}{\rho_+} = -\gamma^2}$$

By total charge conservation $a^2 \rho_+ = \tilde{r}^2 \rho_- \Rightarrow \tilde{r} = a\left(\frac{\rho_+}{\rho_-}\right)^{1/2} \Rightarrow \boxed{\tilde{r} = \frac{a}{\gamma}}$

Plug back into Gauss's Law. For $r < \tilde{r}$

$$E_r = \frac{I}{v\pi a^2}\frac{r}{2\varepsilon_0} - \frac{1}{r\varepsilon_0}\gamma^2\frac{r^2}{2}\frac{I}{v\pi a^2}$$

$$= \frac{I r}{2\pi a^2 \varepsilon_0 v}\left[1 - \gamma^2\right]$$



$$@\ r = \tilde{r}\quad E = \frac{I}{2\pi a \varepsilon_0 v}\left[\frac{1-\gamma^2}{\gamma}\right]$$

b) So, the potential difference is $\left[\frac{1}{2}\tilde{r} + \frac{1}{2}(a - \tilde{r})\right]\frac{I}{2\pi a \varepsilon_0 v}\left[\frac{1-\gamma^2}{\gamma}\right]$
(just do integral of the plot)

$$\Rightarrow \Delta V = \frac{I}{4\pi \varepsilon_0 v}\,\frac{1-\gamma^2}{\gamma}$$

$\boxed{P4}$ $$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5}$$ $$\vec{p} = q\vec{x}$$ $$\ddot{\vec{p}} = q\vec{a}$$

$$= \frac{1}{16\pi^2\varepsilon_0}\frac{|\hat{r}\times(\vec{u}\times\ddot{\vec{p}})|^2}{(\hat{r}\cdot\vec{u})^5}$$ $$\vec{u} = c\hat{r} - \vec{v} = c\hat{r}$$

(non relativistic)

$$\vec{p} = -\frac{V_0 e}{\omega_0}(\sin\omega_0 t, -\cos\omega_0 t, 0)$$

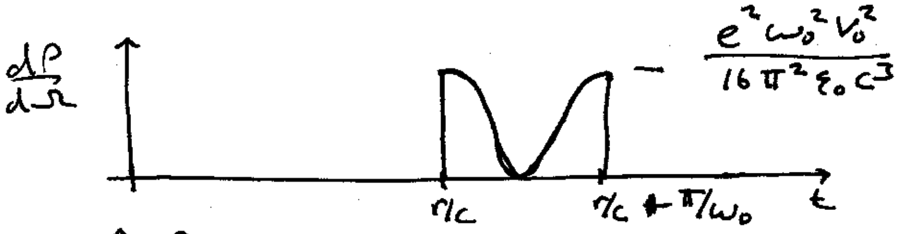
$$\ddot{\vec{p}} = eV_0\omega_0(\sin\omega_0 t, -\cos\omega_0 t, 0) \quad 0<t<\pi/\omega_0$$
$$= 0 \quad t<0,\ t>\pi/\omega_0$$

a) For x direction

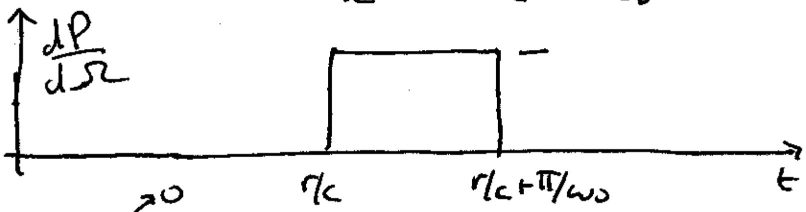
$$\hat{r}\times\ddot{\vec{p}} = \hat{z}(-\cos\omega_0 t_r)\,e\omega_0 V_0$$
$$(\hat{r}\cdot\vec{u})^5 = c^5$$

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0 c^5}\,c^2 e^2\omega_0^2 V^2\cos^2\omega_0(t - r/c)$$ but this happens at r/c

$\Rightarrow$



b) For z direction



c) $$P = \frac{\mu_0}{6\pi c}\left(\ddot{p}^2 - \left|\frac{\cancel{\vec{v}\times\vec{a}}}{c}\right|^2\right)$$ $\rightarrow 0$ $\quad \vec{v}\perp\vec{a}$

$$= \frac{\mu_0}{6\pi c}e^2\omega_0^2 V_0^2 = \frac{\text{Energy}}{\text{time}}$$

It radiates for π/ω_0 seconds $\Rightarrow$ $$\boxed{E_{rad} = \frac{\mu_0 e^2 V_0^2\omega_0}{6c}}$$