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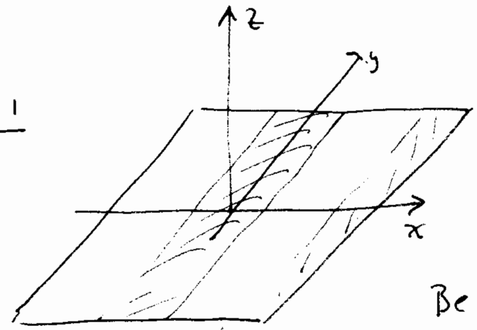
PHY304 FINAL SOLNS

May 13 '05



Problem 1

$\sigma = \sigma_0 \cos(kx)$

$\nabla^2 V = 0$

Because of Y symmetry: $V = XZ$:

use separation of variables

$$\Rightarrow \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-l^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{m^2} = 0 \qquad m = \pm l$$

The solution is then $V = \sum_l (A_l \sin lx + B_l \cos lx)(C_m e^{-mz} + D_m e^{+mz})$

Since $V \to 0$ as $z \to \infty$ $D_m = 0$; we take $m = l$

$$V = \sum_l (A_l' \sin lx + B_l' \cos lx)\, e^{-\cancel{k}z} \qquad A_l' = A_l C_l$$

Boundary condition: $\varepsilon_0\left(-\frac{\partial V}{\partial z}\Big|_a + \frac{\partial V}{\partial z}\Big|_b\right) = \sigma_0 \cos kx$

By symmetry $2\varepsilon_0 \frac{\partial V}{\partial z}\Big|_a = -\sigma_0 \cos kx$

$$= 2\varepsilon_0 \sum_l (A_l' \sin lx + B_l' \cos lx)(-\cancel{l}) = -\sigma_0 \cos kx$$

$$\Rightarrow A_l' = 0 \qquad l = k \Rightarrow B_l' = \frac{\sigma_0}{2k\varepsilon_0}$$

$$\Rightarrow V = \frac{\sigma_0 \cos kx}{2k\varepsilon_0} e^{-kz}$$

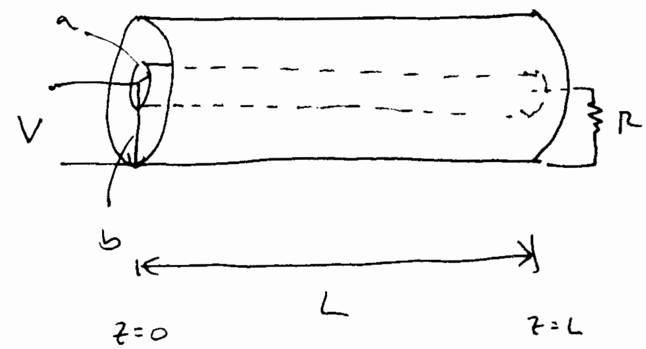
$$\vec{E} = -\vec{\nabla} V = -\frac{\sigma}{2k\varepsilon_0}\left(-\sin kx(k),\ 0,\ \cos kx(-k)\right) e^{-kz}$$

a) $$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\sin kx, 0, \cos kx)\, e^{-kz}$$

b) $$\vec{E} = \frac{\sigma}{2\varepsilon_0} e^{-kz_0}\,\hat{z} \qquad \vec{p} = \alpha\vec{E} = \frac{\sigma\alpha}{2\varepsilon_0} e^{-kz_0}\,\hat{z}$$

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} = -k\alpha\left(\frac{\sigma}{2\varepsilon_0}\right)^2 e^{-2kz_0}\,\hat{z}$$ It's attracted to the plate.

Problem 2 (From the homework!)



ρ = radial coordinate

$V(z) = V_0 - I\underbrace{\left(\frac{1}{\sigma\pi a^2}\right)}_{\text{Resistance/length}} z$

To get the field solve Laplace's equation with:

$V = -\frac{I}{\sigma\pi a^2} z \underbrace{f(\rho)}_{\text{some function of } \rho} + V_0$ for the region between the conductors.

a) $\nabla^2 V = 0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} f(\rho)\right) \Rightarrow \frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) = 0 \Rightarrow \rho\frac{\partial f}{\partial\rho} = \alpha$

$\Rightarrow f = \alpha\ln\rho + \beta$ We can ignore the overall const.

At $\rho = b$ $V = 0$ $\Rightarrow$ $\beta = -\alpha\ln b$

At $\rho = a$ $V = \frac{-I}{\sigma\pi a^2} z\,\alpha\ln(a/b)$ $\Rightarrow$ $\alpha = \frac{1}{\ln a/b}$

$\Rightarrow V = -\frac{Iz}{\pi a^2\sigma}\frac{\ln\rho/b}{\ln a/b}$

Thus $\rho < a$ $\vec{E} = \frac{I}{\pi a^2\sigma}\hat{z}$ $\vec{B} = \frac{\mu_0}{2\pi\rho}\frac{\rho^2 I}{a^2}\varphi$

$a < \rho < b$ $\vec{E} = -\vec{\nabla}V = \left(-\frac{\partial V}{\partial\rho}, -\frac{1}{\rho}\frac{\partial V}{\partial\varphi}, -\frac{\partial V}{\partial z}\right)$

$= \frac{Iz}{\pi a^2\sigma\ln(a/b)}\frac{1}{\rho}\hat{\rho} + \frac{I}{\pi a^2\sigma}\frac{\ln(\rho/b)}{\ln(a/b)}\hat{z}$

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\varphi}$

b) Poynting flux is

$a<\rho<b \qquad \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$

$$= \frac{I}{\pi a^2 \sigma \ln(a/b)}\left[\frac{I}{2\pi}\right]\left[\frac{z(\hat{\rho}\times\hat{\varphi})}{\rho^2} + \frac{\ln(\rho/b)}{\rho}(\hat{z}\times\hat{\varphi})\right]$$

$$= \frac{I^2}{2\pi^2 a^2 \sigma}\left[\frac{z}{\rho^2}\hat{z} - \frac{\ln(\rho/b)}{\rho}\hat{\rho}\right]\frac{1}{\ln(a/b)}$$

$\rho<a \qquad \vec{S} = \frac{1}{\mu_0}\frac{I}{\pi a^2\sigma}\frac{\mu_0}{2\pi}\frac{\rho I}{a^2}\hat{z}\times\hat{\varphi}$

$$= \frac{-I^2}{2\pi^2 a^2\sigma}\left[\frac{\rho}{a^2}\right]\hat{\rho}$$

$\vec{p}_{em} = \mu_0\varepsilon_0\int\vec{S}\,dV$ momentum in radial direction cancels out.

$$= \frac{\mu_0\varepsilon_0 I^2}{2\pi^2 a^2\sigma}\left[\int_a^b \frac{z}{\rho^2}\rho\,d\rho\,d\theta\,dz\right]\frac{1}{\ln(a/b)}$$

$$\vec{p}_{em} = \frac{\mu_0\varepsilon_0 I^2 L^2}{2\pi a^2\sigma}\hat{z}$$

Problem 3

a) $m^*\dot{\vec{v}} = e^*\vec{E}$; $\vec{J} = n e^* \vec{v}$

$$\Rightarrow \dot{\vec{J}} = \underbrace{\frac{n(e^*)^2}{m^*}}_{\alpha}\vec{E}$$

b) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\vec{\nabla}\times\dot{\vec{B}} = \mu_0\dot{\vec{J}} = \mu_0\alpha\vec{E}$

Take curl $\vec{\nabla}(\vec{\nabla}\cdot\dot{\vec{B}}) - \nabla^2\dot{\vec{B}} = \mu_0\alpha\vec{\nabla}\times\vec{E}$

$-\nabla^2\dot{\vec{B}} = \mu_0\alpha(-\dot{\vec{B}})$

$\Rightarrow \nabla^2\dot{\vec{B}} = \mu_0\alpha\dot{\vec{B}}$

Integrate $\nabla^2(\vec{B}(\vec{r},t) - \vec{B}(\vec{r},0)) = \mu_0\alpha(\vec{B}(\vec{r},t) - \vec{B}(\vec{r},0))$

We are told that $\vec{B}(\vec{r},t) = \vec{B}(z,t)$

$\vec{B}(\vec{r},0) = \vec{B}_0(z)$

$$\Rightarrow \frac{d^2}{dz^2}\left(\vec{B}(z,t) - \vec{B}_0(z)\right) = \mu_0\alpha\left(\vec{B}(z,t) - \vec{B}_0(z)\right)$$

$$\Rightarrow \vec{B}(z,t) = \vec{B}_0(z) + \vec{U}(t)\,e^{-z/\lambda} \qquad \lambda = (\mu_0\alpha)^{-1/2}$$

c)

$$\left[\frac{4\pi\times10^{-7}\; 1.5\times10^{28}\,m^{-3}\,(3.2\times10^{-19})^2}{2\cdot 9.11\times10^{-31}\,Kg}\right]^{-1/2}$$

$\Rightarrow \lambda = 3\times10^{-8}\,m$

Problem 4 The transmitted power is absorbed, thus we can compute the transmission and set that equal to α which is also ε. One must recall that an extra factor of $\tilde{n}$ must be included (pg 386). It is easiest to stay in vacuum and compute $R = r^* r = 1 - \alpha = 1 - \varepsilon$

For a good conductor that is non magnetic $\mu = \mu_0$; $\sigma \gg \varepsilon_0 \omega$

$$\tilde{n} = \frac{c}{\omega}\tilde{k} = \frac{c}{\omega}\left(\mu \varepsilon \omega^2 + i \sigma \mu \omega\right)^{1/2}$$

$$\Rightarrow \frac{c}{\omega}\mu_0^{1/2}\omega \varepsilon_0^{1/2}\left(\varepsilon_r + \underbrace{\frac{i\sigma}{\varepsilon_0 \omega}}_{\text{large}}\right)^{1/2} = \left(\varepsilon_r + i\frac{\sigma}{\varepsilon_0 \omega}\right)^{1/2}$$

This is almost a pure imaginary since $\sigma/\varepsilon_0\omega \gg \varepsilon_r$

$$\tilde{n} = \left(\frac{\sigma}{\varepsilon_0 \omega}\right)^{1/2}(1+i)^{1/2} = \left(\frac{\sigma}{\varepsilon_0 \omega}\right)^{1/2}\frac{1+i}{\sqrt{2}} = A(1+i) \qquad A = \left(\frac{\sigma}{2\varepsilon_0 \omega}\right)^{1/2}$$

$$r = \frac{1 - A(1+i)}{1 + A(1+i)} \qquad r^* r = \frac{1 - A(1+i)}{1 + A(1+i)}\,\frac{1 - A(1-i)}{1 + A(1-i)}$$

$$= \frac{1 - 2A + 2A^2}{1 + 2A + 2A^2} = \frac{1 - \frac{1}{A}}{1 + \frac{1}{A}} = 1 - \frac{2}{A} = 1 - \left(\frac{8\varepsilon_0 \omega}{\sigma}\right)^{1/2}$$

$$\Rightarrow \varepsilon = \left(\frac{8\varepsilon_0 \omega}{\sigma}\right)^{1/2}$$

Problem 5

a) $t<0$ there is no acceleration and thus no radiated power $\boxed{0}$

b)
$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5}$$

$$\vec{u} = c\hat{r} - \vec{v}$$

$$\vec{u}\times\vec{a} = c\,\hat{r}\times\vec{a} - \cancel{\vec{v}\times\vec{a}}^{\,0} = c\,\hat{r}\times\vec{a}$$

$$= \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{|\hat{r}\times(\hat{r}\times\vec{a})|^2}{(c-\hat{r}\cdot\vec{v})^5}$$

$$\hat{r}\cdot\vec{a} = -a\cos\theta$$

$$\hat{r}\cdot\vec{v} = v\cos\theta$$

note: sign of a does not enter

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

c) At $\theta = 0$ there is no radiation for a very small detector.

d)
$$E = \int_0^\tau P\,dt = \int_0^\tau \frac{\mu_0 q^2 \gamma^6}{6\pi c} a^2\,dt$$

Typo in formula sheet

OK to leave in this form.