Physics 304 Final Exam May 13, 2005

180 minutes. Closed book. You may use a calculator.

Write your answers in an exam book. Please be sure to put your name on the front of the book! Show your work, explain your reasoning, and please box your final answers.

There are five problems with point values indicated. The total points value is 100.

Good luck.

After you have completed the exam, write and sign the honor pledge on the front of the exam book: **"I pledge my honor that I have not violated the Honor Code during this examination."**

Problem 1. [20 points] A thin sheet of material with $\epsilon_r = 1$ lies in the xy plane and carries a periodic surface charge density given by $\sigma(x) = \sigma_0 \cos(kx)$.

(a)What is the electric field in the region $z > 0$? [Hint: compute the potential first using $\nabla^2 V = 0$ and separation of variables, otherwise a tricky integral is encountered.]

(b)A small neutral object of polarizability α is placed at $\mathbf{r} = (0, 0, z_0)$. What is the force (magnitude and direction) on it?

Problem 2. [20 points] A long straight coaxial cable of length L has an inner conductor radius a and an outer conductor radius b. Assume the insulating region between the conductors has the free-space values of ϵ_0 and μ_0. On one end of the cable, $z = L$, a load resistor R is attached between the central and outer conductors. At the other end, $z = 0$, a battery of voltage V is connected across the conductors. A steady current flows in the conductors. The current may be taken as uniformly distributed throughout the inner conductor, which has conductivity σ. The resistance of the outer conductor can be neglected; the current there flows through a thin surface sheet of radius b.

(a)What is the electric field for $r < b$? You may neglect fringing fields.

(b)What is the electromagnetic momentum in the coax?

Problem 3. [20 points] A superconductor can be described as a neutral material with current carrying "particles" called Cooper pairs (of electrons), of mass $m^* = 2m_e$, charge $e^* = 2e$ and number density n. These Cooper pairs can be accelerated by an electric field according to Newton's second law, $m^*\ddot{\mathbf{x}} = e^*\mathbf{E}$ (we assume that the Lorentz force acting on them can be neglected).

(a) Show that the electric current density $\mathbf{J}$ due to the Cooper pairs obeys the equation $\dot{\mathbf{J}} = \alpha\mathbf{E}$ where α is a constant. What is α in terms of m^*, e^* and n?

(b) Consider a large block of lead that fills out the region $z > 0$ and has a static magnetic field $\mathbf{B}_0(z)$ inside. At time $t = 0$, it is made a superconductor by cooling. Show that for $t > 0$, any magnetic field $\mathbf{B}(z, t)$ is the sum of $\mathbf{B_0}(z)$ and a part that falls off exponentially in z as $\exp(-z/\lambda)$. What is λ in terms of fundamental constants and m^*, e^* and n? You may assume that the electric field varies very slowly in time, so Ampere's law holds inside the superconductor.

(c) For lead $n = 1.5 \times 10^{22}$ cm^{-3}. What is the numerical value of λ? [Numerical values in the formula sheet may be useful.]

Problem 4. [20 points] The Fresnel equations for normal-incidence electric fields from vacuum onto a material of complex index $\tilde{n}$ are given by:

$$r = \frac{\tilde{E}_{ref}}{\tilde{E}_{inc}} = \frac{1-\tilde{n}}{1+\tilde{n}} \qquad\qquad t = \frac{\tilde{E}_{trans}}{\tilde{E}_{inc}} = \frac{2}{1+\tilde{n}} \tag{1}$$

Recall that the complex wave vector is given by $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$ where σ is the conductivity. It can be shown that an object's ability to absorb radiation (absorptivity, α) is equal to its ability to emit radiation (emissivity, e). This is known as Kirchoff's Law. For a perfect absorber all incident radiation is absorbed and $\alpha = 1$. For a perfect reflector no incident radiation is absorbed and $\alpha = 0$. What is the emissivity of a good non-magnetic conductor of conductivity σ for normal incidence?

Problem 5. [20 points] At $t = 0$ a charge moving at a relativistic velocity $\mathbf{v} = v_0\hat{x}$ suddenly decelerates over a short distance with acceleration $\mathbf{a} = -a_0\hat{x}$ until it stops. The deceleration starts at $x_0 = 0$.

(a)What is the radiated power before ($t < 0$) the deceleration?

(b)What is the power radiated into a patch of area $r^2 d\Omega$ in direction (θ, ϕ) during the deceleration?

(c)A small perfectly absorbing detector of solid angle $d\Omega <<< \pi$ sr is set up a long distance away at $\mathbf{r} = (x_0, 0, 0)$. How much power does it absorb as a function of time?

(d)What is the total energy radiated?

$e = 1.6 \times 10^{-19}$ C $\quad m_e = 9.1 \times 10^{-31}$ kg $\quad c = 9 \times 10^{8}$ m/s $\quad \epsilon_0 = 8.85 \times 10^{-12}$ F/m
$\mu_0 = 4\pi \times 10^{-7}$ H/m

Possible useful mathematics:

Cartesian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Cylindrical: $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

It may help to recall that the general solution to $d^2y/dz^2 = k^2 y$ is $y = A\exp(kz) + B\exp(-kz)$. and the general solution to $d^2y/dz^2 = -k^2 y$ is $y = A\sin(kz) + B\cos(kz)$.

$J = \sigma E = \rho v \qquad \tilde{n} = \frac{c\tilde{k}}{\omega} = \tilde{\epsilon}^{1/2} \qquad v = \frac{\omega}{k} = \lambda\nu$

$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{F} = (\mathbf{p} \cdot \vec{\nabla})\mathbf{E} \qquad \mathbf{P} = \epsilon_0 \int \mathbf{E} \times \mathbf{B}\, d\tau \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$

In the following radiation formulas, the $\hat{r}$ is the "$\hat{r}$ slash" used in class.

$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{r}\times(\mathbf{u}\times\mathbf{a})|^2}{(\hat{r}\cdot\mathbf{u})^5} \qquad P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}(a^2 - |\frac{\mathbf{v}\times\mathbf{a}}{c}|^2) \qquad \mathbf{u} = c\hat{\mathbf{r}} - \mathbf{v}$

Electric field boundary conditions:

$E^{\parallel}_{above} = E^{\parallel}_{below}$

$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$

$V_{above} = V_{below}$

$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$

Magnetic field boundary conditions:

$B^{\perp}_{above} = B^{\perp}_{below}$

$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$

$\mathbf{A}_{above} = \mathbf{A}_{below}$

$\frac{\partial \mathbf{A}_{above}}{\partial n} - \frac{\partial \mathbf{A}_{below}}{\partial n} = -\mu_0 \mathbf{K}$