Midterm Solutions

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Problem 1.

First lets see what the boundary condition for the potential is at infinity

$$V \rightarrow -E_o x = -E_o r cos\phi$$

in order to have a constant external field at infinity. Since our only boundary condition has only $cos\phi$ only the $k = 1$ terms are going to survive in the expansion of the potential. In the expansion of the potential one cannot also have the ϕ term and the logarithmic term since we want a single valued and smooth potential at 0 and ∞. The last requirement also excludes the terms r^k for $r > R$ and r^{-k} for $r < R$. So finally just by symmetry arguments we must have

$$V(r < R) = arcos\phi$$

$$V(r > R) = -E_o r cos\phi + \frac{b}{r} cos\phi$$

Continuity at $r = R$ gives

$$aR = -E_o R + \frac{b}{R}$$

Continuity of the D field at r=R (since there are no free charges) gives

$$\epsilon_o \frac{\partial V(r > R)}{\partial r}|_{r=R} = \epsilon_r \epsilon_o \frac{\partial V(r < R)}{\partial r}|_{r=R}$$

$$a\epsilon_r = -E_o - \frac{b}{R^2}$$

The last two equations give

$$a = -\frac{2E_o}{1+\epsilon_r}, b = R^2 E_o \frac{\epsilon_r - 1}{\epsilon_r + 1}$$

and the answer is

$$V(r < R) = -\frac{2E_o}{1+\epsilon_r} r cos\phi$$

$$V(r > R) = -E_o r cos\phi + E_o \frac{R^2}{r} \frac{\epsilon_r - 1}{\epsilon_r + 1} cos\phi$$

Notice that the field inside the cylinder is uniform as in the case of the sphere.

There is also a more fun way to solve this problem as follows. First convince yourself that a surface charge $\sigma = kcos\phi$ on the cylinder gives a constant electric field inside the cylinder with $E = -\frac{k}{2\epsilon_o}$. Outside the potential is given by $V = \frac{kR^2}{2r\epsilon_o}cos\phi$. Now do the following. The external field produces some polarization, this induces a surface bound charge of the form $\epsilon_o(\epsilon_r - 1)E_o cos\phi$. This bound charge is going to produce a new uniform field from the previous argument which will produce a new surface charge and so on. In the end we have to add all these fields. It is easy to see that the second term for example is for example $E_1 = -\frac{\epsilon_r - 1}{2}E_o$ and $E_2 = -\frac{\epsilon_r - 1}{2}E_1$ and so on. So the total electric field is

$$E = \sum_{n=o}^{\infty} E_n = \sum_{n=0}^{\infty}(-1)^n(\frac{\epsilon_r - 1}{2})^n E_o = \frac{1}{1 + \frac{\epsilon_r - 1}{2}}E_o = \frac{2}{\epsilon_r + 1}E_o$$

by summing the geometric series . The potential outside can be found by summing all the bound charges so that

$$\sigma_{total} = \epsilon_o(\epsilon_r - 1)\sum_{n=0}^{\infty} E_n cos\phi = \epsilon_o(\epsilon_r - 1)\sum_{n=0}^{\infty}(-1)^n(\frac{\epsilon_r - 1}{2})^n cos\phi = 2\epsilon_o E_o\frac{\epsilon_r - 1}{\epsilon_r - 1}cos\phi$$

So the total potential outside due to the cylinder is given by

$$V(r > R) = \frac{\sigma_{total}R^2}{2r\epsilon_o} = E_o\frac{R^2}{r}\frac{\epsilon_r - 1}{\epsilon_r + 1}cos\phi$$

as before.

Problem 2 .

We know that for large r the first non zero term is the dipole term in the multipole expansion of the vector potential. We just have to compute the total magnetic dipole moment of the rotating disc and plug it in the formula for the vector potential. We can think of the disc as many circular loops each having a small dipole moment $d\vec{m} = \pi r^2 dI\hat{k}$ and $dI = \frac{dQ}{dt} = \sigma\frac{rdrd\phi}{dt} = \sigma\omega rdr$. So the total magnetic dipole moment is

$$m_{t\vec{o}tal} = \int d\vec{m} = \int_0^R \sigma\omega\pi r^3 dr\hat{k} = \frac{\pi\sigma\omega R^4}{4}\hat{k} = \frac{\pi\sigma\omega R^4}{4}\hat{k}$$

So the vector potential is

$$\vec{A} = \frac{\mu_o}{4\pi}\frac{\vec{m}_{total} \times \hat{r}}{r^2} = \frac{\mu_o\sigma\omega R^4}{16r^2}\hat{\phi}$$

and the magnetic field is that of a dipole

$$\vec{B} = -\frac{\mu_o\sigma\omega R^4}{16r^3}\hat{k}$$

One can use for example (5.86) with $\theta = \pi/2$.

One can also try to calculate immediately the vector potential and the magnetic field. Lets do it only for the vector potential.

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{K}}{\boldsymbol{r}} da'$$

Lets put the observation point on the x- axis . $\vec{K}$ is given by

$$\vec{K} = \sigma\omega r'\hat{\phi} = \sigma\omega r'(-sin\phi'\hat{x} + cos\phi'\hat{y})$$

and $\frac{1}{\boldsymbol{r}}$ can be expanded as follows

$$\frac{1}{\boldsymbol{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'cos\phi'}} \simeq \frac{1}{r}(1 + \frac{r'}{r}cos\phi')$$

This depends on the fact that the observation point is on the x axis. So finally

$$\vec{A} = \frac{\mu_o}{4\pi}\int_0^{2\pi}\int_0^R \sigma\omega r'(-sin\phi'\hat{x} + cos\phi'\hat{y})\frac{1}{r}(1 + \frac{r'}{r}cos\phi')r'dr'd\phi'$$

The only term that survives the ϕ' integration is the $cos^2\phi'$ term . So

$$\vec{A} = \frac{\mu_o\sigma\omega}{4\pi r^2}\int_0^{2\pi}\int_0^R cos^2\phi' r'^3 dr'd\phi'\hat{y} = \frac{\mu_o\sigma\omega R^4}{16r^2}\hat{y}$$

On the x axis $\hat{\phi} = \hat{y}$ so the two results agree. One can similarly try to calculate the magnetic field using Biot-Savart law but we will not do it here.

Problem 3.

Lets set up a coordinate system such that $\hat{z}$ is the direction normal to the conducting plane, $\hat{x}$ is pointing out of the page and $\hat{y}$ is "on" the page.In this coordinate system the dipole is given by

$$\vec{p} = p(cos\theta\hat{z} + sin\theta\hat{y})$$

One can think of the dipole as two charges separated by an infinite distance. So for images we need two charges located at -z. Since the charges should have opposite charges the image charges correspond to a dipole having the same magnitude and a different direction. It is easy to check by the image charges that $\vec{p}_{im}$ is given by

$$\vec{p}_{im} = p(cos\theta\hat{z} - sin\theta\hat{y})$$

The field the image charge is creating is by (3.104)

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{1}{(2z)^3}(3(\vec{p}_{im}\cdot\hat{z})\hat{z} - \vec{p}_{im}) = \frac{1}{4\pi\epsilon_o}\frac{1}{(2z)^3}(2pcos\theta\hat{z} + psin\theta\hat{y})$$

The torque is given by

$$\vec{N} = \vec{p} \times \vec{E} = \frac{1}{4\pi\epsilon_o}\frac{1}{(2z)^3}p^2(cos\theta sin\theta \hat{z} \times \hat{y} + 2cos\theta sin\theta \hat{y} \times \hat{z}) = \frac{p^2}{4\pi\epsilon_o(2z)^3}sin\theta cos\theta \hat{x}$$

and for small θ is pointing upwards.

So points of equilibrium are $\theta = 0, \pi/2, \pi, 3\pi/2$. In order ro find the stable equilibrium we can calculate the total energy of the configuration . This behaves as (4.7) or by just calculating it

$$U = -\frac{1}{2}\vec{E}_{im} \cdot \vec{p} = -\frac{1}{2}\frac{p^2}{4\pi\epsilon_o(2z)^3}(2cos^2\theta + sin\theta) = \frac{1}{2}\frac{p^2}{4\pi\epsilon_o(2z)^3}(-1 - cos^2\theta)$$

The factor of 1/2 comes from the fact that half the space is empty, as in the original image problem of a point charge and a conducting plane.

So the minimum of the potential happens when $cos^2\theta = 1$, i.e. at $\theta = 0, \pi$. If the initial θ is between 0 and $\pi/2$ or between $3\pi/2$ and 2π . it will end its rotation at $\theta = 0$ while if it starts between $\theta = \pi/2$ and $\theta = 3\pi/2$ it will end at π. This can been seen by just sketching the potential $\simeq -cos^2\theta$.

The points $\theta = \pi/2, 3\pi/2$ are unstable equilibrium points, since they correspond to the maximum of the potential.

This conclusion can been reached also by the form of the torque. We see that if θ is in $[0, \pi/2)$ the torque is in the + $\hat{x}$ direction and tends to make the dipole perpendicular to the conducting plane pointing upwards.Similarly for the θ ϵ $(3\pi/2, 2\pi)$ the torque is pointing in the - $\hat{x}$ direction and has the same effect .

With the same arguments we see that is the initial θ ϵ $(\pi/2, 3\pi/2)$ it will end up at $\theta = \pi$.