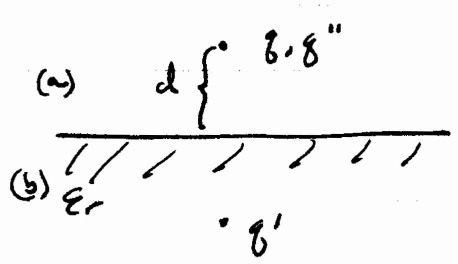
PHY304 MIDTERM 2005 SOLUTIONS

Problem 1

(a)

10 pts



q'' image charge for $z<0$

q' image charge for $z>0$

Note, you can do this problem by taking q & q'' as the source charges for $z<0$, the values for q' & q'' are different than those below but σ_p is the same.

$$z>0 \quad V_a = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left((d-z)^2+\rho^2\right)^{1/2}} + \frac{q'}{\left((d+z)^2+\rho^2\right)^{1/2}}\right]$$

ρ = cylindrical coord on surface.

$$z<0 \quad V_b = \frac{1}{4\pi\varepsilon_0}\left[\frac{q''}{\left((d-z)^2+\rho^2\right)^{1/2}}\right]$$

B.C. At $z=0$ (1) $V_a = V_b$ and (2) $\frac{\partial V_a}{\partial n} - \varepsilon_r \frac{\partial V_b}{\partial n} = 0$

From (1) $q + q' = q''$

From (2)
$$\frac{1}{4\pi\varepsilon_0}\left[\left(-\frac{1}{2}\right)\frac{q\,2(d-z)(-1)}{\left((d-z)^2+\rho^2\right)^{3/2}} + \left(-\frac{1}{2}\right)\frac{q'\,2(d+z)}{\left((d+z)^2+\rho^2\right)^{3/2}}\right]$$
$$= \varepsilon_r \frac{q''}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\frac{2(d-z)(-1)}{\left((d-z)^2+\rho^2\right)^{3/2}}$$

at $z=0$ $\quad -qd + q'd = -\varepsilon_r q'' d$

or $\quad q - q' = \varepsilon_r q''$

$\quad q + q' = q''$

$$\Rightarrow q'' = \frac{2}{\varepsilon_r + 1}\,q \qquad q' = \left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right) q$$

with q & q'' as source for $z<0$ would have found $q' = q'' = \left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right) q$

(b)

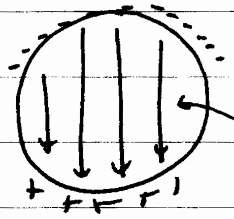
5 pts $\sigma_p = \vec{P}\cdot\vec{n} = P_z = \varepsilon_0(\varepsilon_r - 1)E_z = \varepsilon_0(1-\varepsilon_r)\nabla_z V \quad @\ z=0$

$$\frac{\partial V_b}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\frac{q'' \, 2(d-z)(-1)}{\left((d-z)^2+\rho^2\right)^{3/2}}$$

$$z=0 \rightarrow \frac{1}{4\pi\varepsilon_0}\frac{2q''d}{\varepsilon_r+1}\frac{1}{(d^2+\rho^2)^{3/2}}$$

$$\Rightarrow \sigma_p = \frac{q}{2\pi}\left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right)\frac{d}{(d^2+x^2+y^2)^{3/2}}$$

<u>Problem 2</u> There is a subtle point to this problem. Because $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \neq 0$ in the dielectric, there must be free charges somewhere. This makes imposing BC at $z \to \infty$ problematic. The way to solve the problem is by superposition. As we did in class, and as you did for homework, impose a spherical chunk of dielectric



$\sigma = -P\cos\theta$ by definition.

$\vec{P}$ These are bound charges. What's E?

For $r<R$ $V = Ar\cos\theta$ ($\ell = 1$ survives)

$r>R$ $V = \frac{B}{r^2}\cos\theta$

Continuity @ r=R $AR\cos\theta = \frac{B}{R^2}\cos\theta \Rightarrow B = R^3 A$

Derivative $\left.\frac{\partial V}{\partial r}\right|_{r>R} - \left.\frac{\partial V}{\partial r}\right|_{r<R} = -\frac{\sigma}{\varepsilon_0}$ Total

$$-\frac{2B}{r^3}\cos\theta - A\cos\theta = \frac{P\cos\theta}{\varepsilon_0}$$

at $r=R$: $-\frac{2R^3 A}{R^3} - A = \frac{P}{\varepsilon_0}$ or $3A = \frac{-P}{\varepsilon_0}$

$V = \frac{-P}{3\varepsilon_0} r\cos\theta = -\frac{Pz}{3\varepsilon_0}$ $\vec{E} = -\vec{\nabla} V = \frac{P}{3\varepsilon_0}\hat{z}$

$\Rightarrow \vec{E}_{cavity} = \vec{E}_d + \frac{\vec{P}}{3\varepsilon_0} = \left(\frac{2+\varepsilon_r}{3}\right)\vec{E}_d$

Problem 3 Sphere is uniformly charged $\rho = \frac{3Q}{4\pi R^3}$

a) $\vec{J} = \rho(\vec{\omega}\times\vec{r}) = \rho\omega r\sin\theta\,\hat{\varphi}$ with $\hat{z}$ along $\vec{\omega}$ direction

10 pts

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}\,d\tau = \frac{1}{2}\int rJ\sin\theta\,\hat{z}\,d\tau = \frac{1}{2}\int r\rho\omega r^2\sin^2\theta\,d\tau\,\hat{z}$$

$$= \frac{1}{2}\rho\omega\int r^4\sin^3\theta\,dr\,d\theta\,d\varphi = \frac{1}{2}\rho\omega\frac{R^5\,2\pi}{5}\underbrace{\int_0^\pi \sin^3\theta\,d\theta}_{4/3}\,\hat{z}$$

$$\vec{m} = \frac{4\pi}{15}\rho\omega R^5\hat{z} = \frac{Q\omega R^2}{5}\hat{z}$$

Thus $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^2} = \frac{\mu_0}{15}\rho\omega R^5\frac{\sin\theta}{r^2}\hat{\varphi}$ (Along the current)

b) $\vec{B} = \vec{\nabla}\times\vec{A}$

10 pts

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\hat{\theta}$$

$$= \frac{\mu_0}{15}\rho\omega R^5\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{\theta}\right]$$

$$= \frac{\mu_0}{15}\rho\omega R^5\left[\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

$$= \frac{\mu_0}{15}\rho\frac{\omega R^5}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

On z-axis, $\theta = 0$ and $\hat{r}$ points along $\hat{z}$

Thus $\vec{B} = \frac{2\mu_0}{15}\rho\frac{\omega R^5}{z^3}\hat{z}$